Math 731 Homework 7 (Correction 1)

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1 Problem 17C1

Definition 1.1. *A compact space X is maximal compact provided every strictly finer topology on X is noncompact.*

Proposition 1.2. *A compact space X is maximal compact if and only if every compact subset is closed.*

Proof. ($\Rightarrow$) We establish the contrapositive. To that end, let (X, τ) be a compact space having a compact subset K that is not closed. Furthermore, let $\mathcal{B}$ be a base for the open sets of X.

Now, since K is not closed, $X - K$ is not open. Let τ' be the topology generated by $\tau \cup \{X - K\}$. Evidently, τ' is a strictly finer topology than τ.

We claim next that (X, τ') is compact. To that end, let $\mathcal{C}$ be an open cover (using sets from τ') of X. If $X - K \notin \mathcal{C}$, then only open sets of τ appear in the cover. Since X is compact under τ, we can find a finite subcover of X. Suppose then that $X - K \in \mathcal{C}$. Since K is compact under τ and K is disjoint from $X - K$, we can find a finite subcover of K using elements of $\mathcal{C} - \{X - K\}$. Taking this cover of K together with $X - K$ gives the desired cover of X.

In either case, we conclude that (X, τ) is not maximal, thus establishing the contrapositive.

($\Leftarrow$) We establish the contrapositive. To that end, suppose that (X, τ) is not maximal compact. Let τ' denote a strictly finer compact topology than τ. Since τ' is strictly finer than τ, we can find a closed set F belonging to τ' but not belonging to τ. Since F is a closed (under τ') subset of a compact space, F is compact under τ'. Hence, every open cover of F by open sets of τ' admits a finite subcover. In particular, any open cover of F using only open sets of τ admits a finite subcover, and so F is compact under τ. Hence, under τ, F is a compact subset of X that is not closed, thus establishing the contrapositive. $\square$

2 Problem 17F4

Proposition 2.1. *Let $X_1, X_2, \ldots$ all be first countable. The product $\prod X_n$ is countably compact if and only if each X_n is countably compact.*

Proof. ($\Rightarrow$) Let $\prod X_n$ be countably compact. That is, every countable open cover of $\prod X_n$ admits a finite subcover. For a fixed k, let $\mathcal{C}$ be a countable open cover of X_k. For each open $C \in \mathcal{C}$, let $U_C = \prod U_i$. where $U_k = C$ and $U_i = X_i$ for $i \neq k$. Denote by $\mathcal{D}$ the collection $\{U_C \mid C \in \mathcal{C}\}$. Evidently, $\mathcal{D}$ is a countable open cover of $\prod X_n$, which admits a finite subcover $\mathcal{D}'$ by the countable compactness of $\prod X_n$. It follows that the collection of all U_k from the open sets belonging to $\mathcal{D}'$ (that is, the collection containing the factor corresponding to X_k from each open set in $\mathcal{D}'$) is a finite subcollection of $\mathcal{C}$ covering X_k, and so X_k is countably compact. As k was arbitrary, it follows that each of the product spaces is countably compact.

($\Leftarrow$) As every countably compact space is sequentially compact, it suffices to show that $\prod X_n$ is sequentially compact. To that end, let $\langle x_n \rangle$ be a sequence in the product space. For each i, $\langle x_n(i) \rangle$ is a sequence in X_i. Since X_i is first countable and countably compact, it is sequentially compact. Hence, $\langle x_n(i) \rangle$ has a convergent subsequence $\langle x_{n_k}(i) \rangle$. (This is not itself enough to conclude that the sequence in the product has a convergent subsequence, as, in general, the product of sequentially compact spaces need not be sequentially compact. Hence, I have to make more explicit use of first countability, but I do not see how to do this.) $\square$

3 Problem 17F5

Proposition 3.1. *Continuous images of countably compact spaces are countably compact.*

Proof. Let f be a continuous function from the countably compact space X onto Y and let $\mathcal{C}$ be a countable open cover of Y. By the continuity of f, the set $\{f^{-1}(C) \mid C \in \mathcal{C}\}$ is an open cover of X. Since X is countably compact, $\{f^{-1}(C) \mid C \in \mathcal{C}\}$ admits a finite subcover $\{f^{-1}(C_1), \ldots, f^{-1}(C_n) \mid C_i \in \mathcal{C}\}$ of X. Since f is onto, it follows that the C_i cover Y. Hence, there is a finite subcover of the countable cover $\mathcal{C}$, and therefore Y is countably compact. □

Proposition 3.2. *Closed subspaces of countably compact spaces are countably compact.*

Proof. Let X be countably compact, F a closed subspace of X, and $\mathcal{C}$ a countable open cover of F. For each $C \in \mathcal{C}$, there is an open set $U_C \in X$ such that $U_C \cap F = C$. Now, since F is closed, $X - Y$ is open in X. Hence, $\{X - F\} \cup \{U_C \mid C \in \mathcal{C}\}$ is an open cover of X, which by the countable compactness of X admits a finite subcover $\{X - F\} \cup \{U_1, \ldots, U_n \mid U_i \in X\}$. It follows that $\{U_1 \cap F, \ldots, U_n \cap F\}$ is a finite subcollection of $\mathcal{C}$ covering F. □