Math 730 Homework 7 (Correction 1)

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1 Extra Problem 1

Proposition 1.1. *Let (X, τ) and (Y, σ) be topological spaces, and let $\mathcal{B}$ be a base for τ. The function $f : Y \to X$ is continuous if and only if $f^{\leftarrow}B \in \sigma$ for all $B \in \mathcal{B}$.*

Proof. ($\Rightarrow$) Suppose f is continuous. It follows that

$$\begin{aligned} f \text{ is continuous} &\Rightarrow f^{\leftarrow}U \text{ is open for every open set } U \subset X \\ &\Rightarrow f^{\leftarrow}B \text{ is open for every } B \in \mathcal{B} \qquad \text{(as each } B \in \mathcal{B} \text{ is itself an open set)} \\ &\Rightarrow f^{\leftarrow}B \in \sigma \text{ for every } B \in \mathcal{B} \end{aligned}$$

($\Leftarrow$) Suppose $f^{\leftarrow}B \in \sigma$ for all $B \in \mathcal{B}$. Let U be any open subset of X. Our goal is to show that $f^{\leftarrow}U$ is open in Y, thus establishing the continuity of f.

Since $\mathcal{B}$ is a base for τ, there exists a collection $\mathcal{C} \subset \mathcal{B}$ such that $U = \bigcup\{B \in \mathcal{C}\}$. Now,

$$\begin{aligned} f^{\leftarrow}U &= f^{\leftarrow}\left(\bigcup\{B \in \mathcal{C}\}\right) \\ &= \bigcup_{B \in \mathcal{C}} f^{\leftarrow}B. \end{aligned}$$

By hypothesis, each $f^{\leftarrow}B$ belongs to σ. In other words, each $f^{\leftarrow}B$ is open in Y, and so the union of these sets is open in Y, as well. Thus, we have shown that $f^{\leftarrow}U$ is open in Y for any open $U \subset X$. Therefore, f is continuous. □

2 Problem 6A1

Proposition 2.1. *Let $\boldsymbol{A}$ denote the slotted plane. Any straight line in the plane has the discrete topology as a subspace of $\boldsymbol{A}$.*

Proof. Let τ be the relative topology on any line $L \subset \mathbb{R}^2$ as a subspace of $\mathbf{A}$. Observe first that we can find basic neighborhoods in $\mathbf{A}$ which intersect trivially with L, and so we can construct $\emptyset$. Now, for any point $x \in L$, consider any basic neighborhood $\{x\} \cup A$ in $\mathbf{A}$ where we require that one of the lines removed from the open disk A coincides with L. Under this constraint, $(\{x\} \cup A) \cap L = \{x\}$. Since we can construct any isolated point by intersecting L with some basic neighborhood in $\mathbf{A}$, we can take unions to construct any subset of L. Therefore, any subset of L is open in τ, and so τ is the discrete topology. □

Proposition 2.2. *Let $\boldsymbol{A}$ denote the slotted plane. The topology on any circle in the plane as a subspace of $\boldsymbol{A}$ coincides with its usual topology.*

Proof. Let C be any circle in $\mathbb{R}^2$. Denote the usual topology of C by σ and the relative topology as a subspace of $\mathbf{A}$ by τ. We show that $\sigma = \tau$.

($\subset$) Let O be a basic open set in σ. That is, O an open interval lying on C. Furthermore, $O = G \cap C$, where G is an open Euclidean ball in $\mathbb{R}^2$. As G is an open ball with finitely-many (i.e. zero) lines removed, $G \in \mathbf{A}$. Therefore, $O = G \cap C \in \tau$.

($\supset$) Let O be a basic open set in τ. That is, $O = C \cap A$ for some $A \in \mathbf{A}$. If $A = \emptyset$, then $C \cap \emptyset = \emptyset \in \sigma$. If $A = \mathbb{R}^2$, then $C \cap \mathbb{R}^2 = C \in \sigma$. Otherwise, $C \cap A$ is the finite union of disjoint open intervals lying on C (the open disk of A selects some open interval of C, while the finite number of removed lines subdivides this into a finite number of open subintervals). In this case, we still have $C \cap A \in \sigma$, as desired. □

3 Problem 6A2

Proposition 3.1. *Let $\boldsymbol{B}$ denote the radial plane. The relative topology induced on any straight line as a subspace of $\boldsymbol{B}$ is its usual topology.*

Proof. Let L be any line in the plane. Denote the topology of L inherited from the usual topology on the plane by σ and the relative topology of L as a subspace of $\mathbf{B}$ by τ. We show that $\sigma = \tau$.

($\subset$) Let O be a basic open set of σ. That is,

$$\begin{aligned} O &= (x - \epsilon, x + \epsilon) \\ &= L \cap (x - \epsilon, x + \epsilon) \\ &= L \cap B(x, \epsilon). \end{aligned}$$

Hence, O is an element of τ.

($\supset$) Let O be a basic open set of τ. That is, O is the union of a collection of open line segments centered around some point $x \in \mathbb{R}^2$. If O intersects trivially with L, then $L \cap O = \emptyset \in \sigma$. If x lies on L, O contains an open line segment centered at x coinciding with an open interval of L. Hence, $L \cap O$ is an open interval lying on L, and so $L \cap O \in \sigma$. □

Proposition 3.2. *Let $\boldsymbol{B}$ denote the radial plane. The relative topology on any circle in the plane as a subspace of $\boldsymbol{B}$ is the discrete topology.*

Proof. Let C be any circle in the plane. Denote the relative topology of C as a subspace of $\mathbf{B}$ by τ. For $x \notin C$, we can always find an open ball about x of small enough radius that intersects trivially with C. Hence, $\emptyset \in \tau$. Now, for any $x \in C$, we wish to find an open set $O \in \mathbf{B}$ such that $C \cap O = \{x\}$. We require that O possess an open line segment about x in each direction. We claim that $O = B(x, \epsilon) \setminus C$ suffices, where ϵ is the radius of C. To see this, consider the line passing through x in a given direction. If the line is tangent to x, we have an open neighborhood of x of length 2ϵ. Otherwise, the line is a chord of C, and so intersects C at some point y. In this direction, we have any open neighborhood of length $\epsilon + d(x, y)$. Hence, O is radially open about x and is constructed in such a way that $C \cap O = \{x\}$. By taking unions, we see that any subset of C is open in the relative topology τ, and so τ is the discrete topology. □

4 Problem 6C

Proposition 4.1. *If M is metrizable and $N \subset M$, then the subspace N is metrizable with the topology generated by the restriction of any metric which generates the topology on M.*

Proof. Let τ be the topology on M generated by a metric ρ. Let σ be the relative topology on N and let ρ_N be the restriction of ρ to N. We show that σ is generated by ρ_N.

Let $O \in \sigma$. It must be that $O = N \cap G$ for some $G \in \tau$. Since M is generated by ρ, we know that $G = \bigcup_{x \in G} B_\rho(x, \epsilon_x)$, where $\epsilon_x > 0$ may depend on x. Now,

$$\begin{aligned} O &= N \cap G \\ &= N \cap \bigcup_{x \in G} B_\rho(x, \epsilon_x) \\ &= \bigcup_{x \in G} N \cap B_\rho(x, \epsilon_x) \\ &= \bigcup_{x \in N \cap G} B_{\rho_N}(x, \epsilon_x). \end{aligned}$$

Hence, O is the union of open balls with respect to the metric ρ_N. Therefore, σ is generated by the ρ_N, as desired. □