

Problem 1

Integrability of f on \mathbb{R} does not necessarily imply the convergence of $f(x)$ to 0 as $x \rightarrow \infty$.

a. There exists a positive continuous function f on \mathbb{R} so that f is integrable on \mathbb{R} , but yet $\limsup_{x \rightarrow \infty} f(x) = \infty$.

Proof. Define the function f to take on the value n if $x \in [n, n + \frac{1}{n^3})$ for $n \geq 2$. Elsewhere, the function is zero except for the line segments required to make the function continuous. We define these segments in such a that the graph resembles a sequence of trapezoids with height n and bases of length $\frac{1}{n^3}$ and $\frac{3}{n^3}$.

By construction, f is positive and continuous on \mathbb{R} . Now,

$$\begin{aligned} \int_{-\infty}^{\infty} f dx &= \sum_{n=2}^{\infty} n \left(\frac{1}{n^3} + \frac{3}{n^3} \right) \\ &= \sum_{n=2}^{\infty} \frac{2}{n^2} \\ &< \infty \end{aligned}$$

Hence, f is integrable. The fact that $\limsup_{x \rightarrow \infty} f(x) = \infty$ is also immediate (for any N , there are infinitely-many $x \in (N, \infty)$ such that $f(x) > N$), so the claim is proven. □

b. However, if we assume that f is uniformly continuous on \mathbb{R} and integrable, then $\limsup_{|x| \rightarrow \infty} f(x) = 0$.

Proof. Suppose, for the sake of contradiction, that $\limsup_{x \rightarrow \infty} f(x) = c > 0$ (we consider first only the case where $x \rightarrow +\infty$). Choose some d so that $0 < d < c$. Then, there is a sequence x_1, x_2, \dots with each x_i far apart (to be made precise later) so that $f(x_i) \geq d$ for all i . Choose ϵ_0 so that $0 < \epsilon_0 < d$. Since f is uniformly continuous, there is some $\delta_0 > 0$ so that, for each x_i , $|f(x_i) - f(y)| < \epsilon_0$ for all $y \in N(x_i, \delta_0)$. Since $f(x_i) \geq d > \epsilon_0$, we have that $f(y) > \epsilon_0$ for all $y \in N(x_i, \delta_0)$. Hence, the area contributed by the function over the interval $N(x_i, \delta_0)$ is at least $2\delta_0\epsilon_0$. Now, if we choose the x_i far enough apart so that each of the $N(x_i, \delta_0)$ are disjoint, we have that

$$\int_0^{\infty} f(x) \geq \sum_{n=1}^{\infty} 2\delta_0\epsilon_0 = \infty$$

which contradicts the fact that f is integrable.

We can force the same contradiction when $x \rightarrow -\infty$, and so we conclude that $\limsup_{|x| \rightarrow \infty} f(x) = 0$. □

Problem 2

Suppose $f \geq 0$, and let $E_{2^k} = \{x \mid f(x) > 2^k\}$ and $F_k = \{x \mid 2^k < f(x) \leq 2^{k+1}\}$. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\},$$

and the sets F_k are disjoint.

Proof. Since f is a function, it has a unique output for each input x . Hence, $2^k < f(x) \leq 2^{k+1}$ for a *single* value of k . That is, the F_k are disjoint.

(\subseteq) Let $x \in \bigcup_{k=-\infty}^{\infty} F_k$. Then $2^k < f(x) \leq 2^{k+1}$ for some k , so $f(x) \neq 0$. Since $f \geq 0$, we have that $f(x) > 0$. That is, $x \in \{f(x) > 0\}$.

(\supseteq) Let $x \in \{f(x) > 0\}$. Then $f(x) > 0$, and so $2^k < f(x) \leq 2^{k+1}$ for some k . That is, $x \in \bigcup_{k=-\infty}^{\infty} F_k$. \square

Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$$

if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty$$

Proof. (i \Rightarrow ii) Suppose f is integrable. Let $x \in F_k$ for some k . By definition of F_k , $2^k < f(x)$. Since the F_k are disjoint, it follows that

$$\begin{aligned} \infty &> \int f dx \\ &> \sum_{k=-\infty}^{\infty} \int 2^k \chi_{F_k} dx \\ &= \sum_{k=-\infty}^{\infty} 2^k \int \chi_{F_k} dx \\ &= \sum_{k=-\infty}^{\infty} 2^k m(F_k) \end{aligned}$$

(ii \Rightarrow i) Suppose $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Then $2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Let $x \in F_k$ for some k . By definition of F_k , $2^{k+1} \geq f(x)$. Since the F_k are disjoint, it follows that

$$\begin{aligned} \int f dx &< \sum_{k=-\infty}^{\infty} \int 2^{k+1} \chi_{F_k} dx \\ &= \sum_{k=-\infty}^{\infty} 2^{k+1} \int \chi_{F_k} dx \\ &= \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k) \\ &< \infty \end{aligned}$$

(ii \Leftrightarrow iii) Suppose $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Observe that $E_{2^k} = \bigcup_{n \geq k} F_n$. Since the F_n are disjoint and measurable, it follows that

$$\begin{aligned} m(E_{2^k}) &= m\left(\bigcup_{n \geq k} F_n\right) \\ &= \sum_{n \geq k} m(F_n) \end{aligned}$$

So

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{\infty} \sum_{n \geq k} 2^k m(F_n) \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^n 2^k m(F_n) \\ &= \sum_{n=-\infty}^{\infty} m(F_n) \sum_{k=-\infty}^n 2^k \\ &= \sum_{n=-\infty}^{\infty} 2^{n+1} m(F_n) \\ &= 2 \sum_{n=-\infty}^{\infty} 2^n m(F_n) \end{aligned}$$

Hence, if either of $\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k})$ or $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$ is finite, then the other is also finite. \square

Use this result to verify the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

Then f is integrable on \mathbb{R}^d if and only if $a < d$; also g is integrable on \mathbb{R}^d if and only if $b > d$.

Proof. Let $x \in F_k$. Then

$$\begin{aligned} 2^k &< |x|^{-a} \leq 2^{k+1} \\ 2^{\frac{-k}{a}} &> |x| \geq 2^{\frac{-k-1}{a}} \end{aligned}$$

Hence

$$\begin{aligned} m(B(0; 2^{\frac{-k}{a}})) &> m(F_k) \geq m(B(0; 2^{\frac{-k-1}{a}})) \\ v_d 2^{\frac{-dk}{a}} &> m(F_k) \geq v_d 2^{\frac{-d(k+1)}{a}} \end{aligned}$$

where v_d is the volume of the unit ball. Now

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^0 2^k m(F_k) + \sum_{k=1}^{\infty} 2^k m(F_k)$$

Since $|x| \leq 1$, $m(F_k) \leq v_d$ for all k . Hence

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(F_k) &\leq v_d \sum_{k=-\infty}^0 2^k + \sum_{k=1}^{\infty} 2^k m(F_k) \\ &= 2v_d + \sum_{k=1}^{\infty} 2^k m(F_k) \end{aligned}$$

So it suffices to show that $\sum_{k=1}^{\infty} 2^k m(F_k)$ converges.

$$\begin{aligned} \sum_{k=1}^{\infty} 2^k (v_d 2^{\frac{-d(k+1)}{a}}) &\leq \sum_{k=1}^{\infty} 2^k m(F_k) < \sum_{k=1}^{\infty} 2^k (v_d 2^{\frac{-dk}{a}}) \\ v_d 2^{\frac{-d}{a}} \sum_{k=1}^{\infty} 2^{k(1-\frac{d}{a})} &\leq \sum_{k=1}^{\infty} 2^k m(F_k) < v_d \sum_{k=1}^{\infty} 2^{k(1-\frac{d}{a})} \end{aligned}$$

We see that the upper and lower bounds converge if and only if $0 < a < d$, forcing the convergence of $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$, which in turn implies that f is integrable.

Let $x \in E_{2^k}$. Then

$$\begin{aligned} 2^k &< |x|^{-b} \\ 2^{\frac{-k}{b}} &> |x| \end{aligned}$$

Observe also that E_{2^k} is empty for $k \leq 0$. Hence, $E_{2^k} = B(0; 2^{\frac{-k}{b}}) \setminus B(0; 1)$, and so $m(E_{2^k}) = v_d 2^{\frac{-dk}{b}} - v_d$. Now

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{-1} 2^k m(E_{2^k}) \\ &= \sum_{k=-\infty}^{-1} 2^k (v_d 2^{\frac{-dk}{b}} - v_d) \\ &= v_d \sum_{k=-\infty}^{-1} 2^{k(1-\frac{d}{b})} - v_d \sum_{k=-\infty}^{-1} 2^k \end{aligned}$$

which converges if and only if $b > d$, which in turn implies that f is integrable. □

Problem 3

a. Prove that if f is integrable on \mathbb{R}^d , real-valued, and $\int_E f(x) dx \geq 0$ for every measurable E , then $f(x) \geq 0$ a.e. x .

Proof. Define F to be the set $\{x \mid f(x) < 0\}$. Since f is integrable, f is measurable, and so F is measurable. We claim that $m(F) = 0$.

Since F is measurable, we have that $\int_F f(x)dx \geq 0$ by hypothesis. Now, observe that, for any $n \geq 1$, $n f \chi_F \leq f$. It follows that

$$\begin{aligned} \int n f \chi_F dx &\leq \int f dx \\ n \int_F f dx &\leq \int f dx \\ \int_F f dx &\leq \frac{1}{n} \int f dx \\ \int_F f dx &\leq 0 \end{aligned}$$

Hence, $\int_F f dx = 0$. Since $f(x) < 0$ for all $x \in F$, we conclude that $m(F) = 0$. That is, $f(x) \geq 0$ almost everywhere. □

b. As a result, if $\int_E f(x)dx = 0$ for every measurable E , then $f(x) = 0$ a.e.

Proof. From the first part, we see that $f(x) \geq 0$ almost everywhere. Let G be the set $\{x \mid f(x) > 0\}$. It suffices to show that $m(G) = 0$.

As before, f is measurable, so G is measurable. By hypothesis, we have that $\int_G f dx = 0$. Since $f(x) > 0$ for all $x \in G$, we conclude that $m(G) = 0$. Hence, the set of x so that $f(x) \neq 0$ has measure 0. That is, $f(x) = 0$ almost everywhere. □

Problem 4

a. Let $a_n, b_n \in \mathbb{R}$ such that $a_n \rightarrow a \in \mathbb{R}$. Prove that

$$\liminf(a_n + b_n) = a + \liminf b_n$$

Proof. Since $a_n \rightarrow a$, we have $a = \lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$. Since $\limsup(a_n) = -\liminf(-a_n)$, we have

$$\begin{aligned} \liminf(a_n + b_n) &= -\limsup(-a_n - b_n) \\ &\geq -\limsup(-a_n) - \limsup(-b_n) \\ &= \liminf a_n + \liminf b_n \\ &= a + \liminf b_n \end{aligned}$$

Now, construct a subsequence (b_{n_k}) of (b_n) with $\lim_{k \rightarrow \infty} b_{n_k} = \liminf_{n \rightarrow \infty} b_n$. Let (a_{n_k}) be the subsequence induced by the indices chosen for (b_{n_k}) .

$$\begin{aligned} a + \liminf b_n &= \liminf a_n + \liminf b_n \\ &= \lim a_{n_k} + \lim b_{n_k} \\ &= \lim(a_{n_k} + b_{n_k}) \\ &\geq \liminf(a_n + b_n) \end{aligned}$$

Therefore, $\liminf(a_n + b_n) = a + \liminf b_n$. □

b. Let f, f_n be integrable functions. Assume $f_n(x) \rightarrow f(x)$ a.e. and $\int |f_n| dx \rightarrow \int |f| dx$. Prove that $\int |f_n - f| dx \rightarrow 0$.

Proof. Define the function g_n to be $|f| + |f_n| - |f - f_n|$. Then $g_n \rightarrow 2|f|$ as $n \rightarrow \infty$. By Fatou's lemma

$$\int g dx \leq \liminf \int g_n dx$$

Hence

$$\begin{aligned} 2 \int |f| dx &\leq \liminf \int (|f| + |f_n| - |f - f_n|) dx \\ &= \liminf \left(\int |f| dx + \int |f_n| dx - \int |f - f_n| dx \right) \\ &= \int |f| dx + \liminf \left(\int |f_n| dx - \int |f - f_n| dx \right) \\ &= \int |f| dx + \int |f| dx + \liminf \left(- \int |f - f_n| dx \right) \quad (\text{by part (a)}) \end{aligned}$$

Now,

$$\begin{aligned} 0 &\leq \liminf \left(- \int |f - f_n| dx \right) \\ 0 &\geq \limsup \int |f - f_n| dx \\ &\geq \lim_{n \rightarrow \infty} \int |f - f_n| dx \end{aligned}$$

Therefore, as $n \rightarrow \infty$, $\int |f - f_n| dx \rightarrow 0$.

□