

Math 731 Homework 6 (Correction 1)

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1 Problem 16C1

Definition 1.1. Let \aleph be any cardinal number. A space X is said to have caliber \aleph if, whenever \mathcal{U} is a family of open subsets of X with $|\mathcal{U}| = \aleph$, then there exists a subfamily \mathcal{V} of \mathcal{U} with $|\mathcal{V}| = \aleph$ and $\bigcap \mathcal{V} \neq \emptyset$.

Proposition 1.2. Every separable space has caliber \aleph_1 .

Proof. Let X be a separable space having countable dense subset D . Let \mathcal{U} be a collection of open subsets of X with $|\mathcal{U}| = \aleph_1$. As D is dense in X , each open set of \mathcal{U} contains a point of D . It follows that there exists $x \in D$ belonging to uncountably-many open sets of \mathcal{U} . (Were this not the case, each of the elements of the countable set D appears in only countably-many open sets of \mathcal{U} , which would imply that \mathcal{U} is countable.) Let $\mathcal{V} = \{U \in \mathcal{U} \mid x \in U\}$. Evidently, $|\mathcal{V}| = \aleph_1$ and $x \in \bigcap \mathcal{V}$. Therefore, X has caliber \aleph_1 . \square

2 Problem 16C3

Definition 2.1. We say X satisfies the countable chain condition provided every family of disjoint open subsets of X is countable.

Proposition 2.2. If X has caliber \aleph_1 , then X satisfies the countable chain condition.

Proof. We establish the contrapositive. To that end, suppose that X does not satisfy the countable chain condition. It follows that there is an uncountable family \mathcal{U} of disjoint open sets of X . By the Axiom of Choice, we may assume

that $|\mathcal{U}| = \aleph_1$ (if $|\mathcal{U}| > \aleph_1$, we use the Axiom of Choice to reduce it to a collection of size \aleph_1). Now, since \mathcal{U} is a collection of disjoint sets, it follows that $\bigcap \mathcal{U} = \emptyset$, and so X does not have caliber \aleph_1 . \square

3 Problem 16D3

Proposition 3.1. *A regular space is Lindelöf if and only if each open cover has a countable subcollection whose closures cover (i.e. has a countable dense subsystem).*

Proof. (\Rightarrow) Let X be a Lindelöf space. Any open cover of X admits a countable subcover \mathcal{U} . Evidently, $U \subset \overline{U}$ for all $U \in \mathcal{U}$, and so the set $\{\overline{U} \mid U \in \mathcal{U}\}$ is also a countable cover of X .

(\Leftarrow) Let X be a regular space and let \mathcal{U} be an open cover of X . For each $x \in X$, let U_x be an open set in \mathcal{U} containing x . Since X is regular, there is an open set V_x containing x with $\overline{V_x} \subset U_x$. Let \mathcal{V} be the collection of sets V_x for all $x \in X$. Evidently, \mathcal{V} is also a cover of X . By hypothesis, \mathcal{V} admits a countable subcollection $\mathcal{V}' = \{V_i \mid i \in \mathbb{N}\}$ such that

$$\bigcup_{i \in \mathbb{N}} \overline{V_i} = X.$$

By construction, each $\overline{V_i}$ is contained in some open set (call it U_i) belonging to \mathcal{U} . It follows that the collection $\{U_i \mid i \in \mathbb{N}\}$ is a countable subcollection of \mathcal{U} covering X . Therefore, X is Lindelöf. \square