Math 730 Homework 6

Austin Mohr

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1 Problem 3A2

Proposition 1.1. *If* $A \subset X$*, then the family* τ *of all subsets of* X *which contain* A*, together with the empty set* ϕ*, is a topology on* X*.*

Proof. We show that τ satisfies each of the three properties of topologies on X.

Claim 1. *If* $O_\alpha \in \tau$ *for all* α *belonging to some indexing set* I*, then* $\bigcup_{\alpha \in I} O_\alpha \in \tau$*.*

Proof. Let $O_\alpha \in \tau$ for all $\alpha \in I$. If this collection consists of *only* the empty set, then the union is itself empty, and so the union belongs to τ. Otherwise, we may disregard any occurrence of the empty set in the collection (it contributes nothing to the union). Adopting this convention, we have

$$\begin{aligned} A \subset O_\alpha \text{ for all } \alpha \in I &\Rightarrow A \subset \bigcup_{\alpha \in I} O_\alpha \\ &\Rightarrow \bigcup_{\alpha \in I} O_\alpha \in \tau. \end{aligned}$$

□

Claim 2. *If* $O_i \in \tau$ *for all* $1 \leq i \leq n$*, then* $\bigcap_{i=1}^{n} O_i \in \tau$*.*

Proof. Let $O_i \in \tau$ for $1 \leq i \leq n$. If O_k is the empty set for any k, then the intersection is itself empty, and so the intersection belongs to τ. Otherwise,

$$\begin{aligned} A \subset O_i \text{ for } 1 \leq i \leq n &\Rightarrow A \subset \bigcap_{i=1}^{n} O_i \\ &\Rightarrow \bigcap_{i=1}^{n} O_i \in \tau. \end{aligned}$$

□

Claim 3. *The empty set belongs to* τ *and the set* X *belongs to* τ*.*

Proof. The empty set is included in τ by definition, and it is clear that $A \subset X$, so $X \in$ *tau*. $\square$

Therefore, τ is a topology on X. $\square$

Remark 1.2. *Recall that the interior of a subset B of X is defined as*

$$B^\circ = \bigcup \{O \mid O \text{ open, } O \subset B\}.$$

Now, if $A \not\subset B$, then the only open subset contained in B is $\emptyset$, and so $B^\circ = \emptyset$. Otherwise, B is itself an open set, and so $B^\circ = B$.

Recall that the closure of a subset B of X is defined as

$$\overline{B} = \bigcap \{F \mid F \text{ closed, } B \subset F\}.$$

Observe that

$$\begin{aligned} F \text{ is closed} &\Leftrightarrow F^c \text{ is open} \\ &\Leftrightarrow A \subset F^c \\ &\Leftrightarrow A \cap F = \emptyset. \end{aligned}$$

Now, if $A \cap B \neq \emptyset$, then the only closed subset containing B is X, and so $\overline{B} = X$. Otherwise, B is itself a closed set, and so $\overline{B} = B$.

Remark 1.3. *If $A = \phi$, then every subset of X is open (as every set contains $\emptyset$), so τ is the discrete topology. If $A = X$, then only X and $\emptyset$ are open (as X can only be contained in itself and we define $\emptyset$ to be open), so τ is the indiscrete topology.*

2 Problem 3C

Proposition 2.1. *If A is any subset of a topological space, the largest possible number of sets in the two sequences*

$$\begin{gathered} A, A^-, A^{-c}, A^{-c-}, \cdots \\ A, A^c, A^{c-}, A^{c-c}, \cdots \end{gathered}$$

(where ^{c} denotes complementation and $^-$ denotes closure) is 14. Furthermore, there is a subset of $\mathbb{R}$ that gives 14.

Proof. Let A be any subset of a topological space and denote the interior operation by $^\circ$. In the first sequence, we see that

$$
\begin{aligned}
A^{-c-c-c-} &= (A^{-c})^{-c-c-} \\
&= (A^{c\circ})^{-c-c-} && \text{(since } E^{-c} = E^{c\circ} \text{ for all sets } E\text{)} \\
&= (A^{c\circ-})^{c-c-} \\
&= (A^{c\circ-})^{\circ cc-} && \text{(since } E^{c-} = E^{\circ c} \text{ for all sets } E\text{)} \\
&= A^{c\circ-\circ cc-} \\
&= A^{c\circ-\circ-} && \text{(since } E^{cc} = E \text{ for all sets } E\text{)} \\
&= (A^{c\circ})^{-\circ-} \\
&= (A^{c\circ})^{-} && \text{(since } A^{c\circ} \text{ is open and } G^{-\circ-} = G^{-} \text{ for all \textit{open} sets } G\text{)} \\
&= (A^{-c})^{-} && \text{(since } E^{c\circ} = E^{-c} \text{ for all sets } E\text{)} \\
&= A^{-c-},
\end{aligned}
$$

which already appears on the list. Hence, we can get at most seven sets in this way (including the original set A).

In the second sequence, we have

$$
\begin{aligned}
A^{c-c-c-c-} &= (A^{c})^{-c-c-c-} \\
&= (A^{c})^{-c-} && \text{(by the previous argument),}
\end{aligned}
$$

which already appears on the list. Hence, we can get at most seven new sets in this way (here, we exclude the original set A, as it has alerady been counted). In total, then, there can be at most 14 distinct sets in these sequences.

We exhibit a subset of $\mathbb{R}$ that achieves the bound. Let

$$A = [0,1] \cup (2,3) \cup \{(4,5) \cap \mathbb{Q}\} \cup \{(6,8) - \{7\}\} \cup \{9\}.$$

The first sequence gives

$$
\begin{aligned}
A^{-} &= [0,1] \cup [2,3] \cup [4,5] \cup [6,8] \cup \{9\} \\
A^{-c} &= (-\infty,0) \cup (1,2) \cup (3,4) \cup (5,6) \cup (8,9) \cup (9,\infty) \\
A^{-c-} &= (-\infty,0] \cup [1,2] \cup [3,4] \cup [5,6] \cup [8,\infty) \\
A^{-c-c} &= (0,1) \cup (2,3) \cup (4,5) \cup (6,8) \\
A^{-c-c-} &= [0,1] \cup [2,3] \cup [4,5] \cup [6,8] \\
A^{-c-c-c} &= (-\infty,0) \cup (1,2) \cup (3,4) \cup (5,6) \cup (8,\infty),
\end{aligned}
$$

and the second sequence gives

$$
\begin{aligned}
A^{c} &= (-\infty,0) \cup (1,2] \cup [3,4] \cup \{(4,5) - \mathbb{Q}\} \cup [5,6] \cup \{7\} \cup [8,9) \cup (9,\infty) \\
A^{c-} &= (-\infty,0] \cup [1,2] \cup [3,6] \cup \{7\} \cup [8,\infty) \\
A^{c-c} &= (0,1) \cup (2,3) \cup (6,7) \cup (7,8) \\
A^{c-c-} &= [0,1] \cup [2,3] \cup [6,8] \\
A^{c-c-c} &= (-\infty,0) \cup (1,2) \cup (3,6) \cup (8,\infty) \\
A^{c-c-c-} &= (-\infty,0] \cup [1,2] \cup [3,6] \cup [8,\infty) \\
A^{c-c-c-c} &= (0,1) \cup (2,3) \cup (6,8),
\end{aligned}
$$

giving a total of 14 distinct sets, thus meeting the upper bound. $\square$

3 Problem 4A3

Definition 3.1. *The Sorgenfrey line, denoted* $\boldsymbol{E}$*, is the real line with the topology in which basic neighborhoods of* x *are the sets* $[x, z)$ *for* $z > x$*.*

Proposition 3.2. *The closure in* $\boldsymbol{E}$ *of* $\mathbb{Q}$ *is* $\mathbb{R}$*.*

Proposition 3.3. *The closure in* $\boldsymbol{E}$ *of* $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ *is itself together with* $\{0\}$*.*

Proposition 3.4. *The closure in* $\boldsymbol{E}$ *of* $\{-\frac{1}{n} \mid n \in \mathbb{N}\}$ *is itself.*

Proposition 3.5. *The closure in* $\boldsymbol{E}$ *of the integers is itself.*

4 Problem 4B1

Definition 4.1. *Let* Γ *denote the closed upper half-plane* $\{(x, y) \mid y \geq 0\}$ *in* $\mathbb{R}^2$*. For each point in the open upper half-plane, basic neighborhoods will be the usual open disks (with the restriction, of course, that they be taken small enough to lie in* Γ*). At the points* z *on the* x*-axis, the basic neighborhoods will be the sets* $\{z\} \cup A$*, where* A *is an open disk in the upper half-plane, tangent to the* x*-axis at* z*. This collection of basic neighborhoods is known as the Moore plane.*

Proposition 4.2. *The Moore plane gives a topology on* Γ*.*

Proof. Recall that, if a collection $\mathcal{B}_x$ of subsets of X is assigned to each $x \in X$ so as to satisfy

V-a) if $V \in \mathcal{B}_x$, then $x \in V$,

V-b) if $V_1, V_2 \in \mathcal{B}_x$, then there is some $V_3 \in \mathcal{B}_x$ such that $V_3 \subset V_1 \cap V_2$,

V-c) if $V \in \mathcal{B}_x$, there is some $V_0 \in \mathcal{B}_x$ such that, for any $y \in V_0$, there is some $W \in \mathcal{B}_y$ with $W \subset V$,

and if we define a set G to be "open" if and only if it contains a basic neighborhood of each of its points, then the result is a topology on X in which $\mathcal{B}_x$ is a neighborhood base at x, for each $x \in X$.

Let $\mathcal{B}_x$ denote the neighborhood base of a point $x \in \Gamma$ given by the Moore plane. We proceed by verifying that $\mathcal{B}_x$ satisfies each of the above properties for each $x \in \Gamma$.

Claim 4. *If* $V \in \mathcal{B}_x$*, then* $x \in V$*.*

Proof. Let $V \in \mathcal{B}_x$. Either V is an open ball centered at x, or V is an open ball tangent to x together with x itself. In either case, we see that $x \in V$, as desired. $\square$

Claim 5. *If* $V_1, V_2 \in \mathcal{B}_x$*, then there is some* $V_3 \in b_x$ *such that* $V_3 \subset V_1 \cap V_2$*.*

Proof. Let V_1 and V_2 belong to $\mathcal{B}_x$. We consider two cases.

Case x lies on the real line

As x lies on the real line, it must be that V_1 and V_2 are both tangent to x. Furthermore, we have that one is contained in the other. Without loss of generality, let $V_1 \subset V_2$. Choosing V_3 to be V_1, we have $V_3 = V_1 \in \mathcal{B}_x$ and $V_3 = V_1 \subset V_1 \cap V_2$, as desired.

Case x lies strictly in the upper half-plane

As x lies strictly in the upper half-plane, it must be that V_1 and V_2 are both open balls centered at x. Furthermore, we have that one is contained in the other. Without loss of generality, let $V_1 \subset V_2$. Choosing V_3 to be V_1, we have $V_3 = V_1 \in \mathcal{B}_x$ and $V_3 = V_1 \subset V_1 \cap V_2$, as desired. $\square$

Claim 6. *If $V \in \mathcal{B}_x$, there is some $V_0 \in b_x$ such that, for any $y \in V_0$, there is some $W \in b_y$ with $W \subset V$.*

Proof. Let $V \in \mathcal{B}_x$. We consider two cases.

Case x lies on the real line

As x lies on the real line, it must be that V is tangent to x. Choose V_0 to be V. If y is chosen to be x, then taking $W = V$ suffices. Otherwise, y lies strictly inside the open ball $V - \{x\}$, and so there will always be a smaller open ball centered at y and contained in V (which we will take to be W).

Case x lies strictly in the upper half-plane

As x lies strictly in the upper half-plane, it must be that V is an open ball centered at x. Choose V_0 to be V. As V is open, for any $y \in V_0 = V$, there is a smaller open ball centered at y and contained in V (which we will take to be W). $\square$

Therefore, the Moore plane gives a topology on Γ. $\square$