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Math 701
Homework 5

Problem 17

Prove that if $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{K}$ are finite Abelian groups and $\mathbf{G} \times \mathbf{H} \cong \mathbf{G} \times \mathbf{K}$, then $\mathbf{H} \cong \mathbf{K}$.

*Proof.* Since $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{K}$ are finite Abelian groups, they are each isomorphic to some direct product of cyclic groups of prime power order.

Observe that, since $\mathbf{G} \times \mathbf{H} \cong \mathbf{G} \times \mathbf{K}$ and all the groups are finite,

$$|\mathbf{G}||\mathbf{H}| = |\mathbf{G} \times \mathbf{H}| = |\mathbf{G} \times \mathbf{K}| = |\mathbf{G}||\mathbf{K}|$$

Hence, $|\mathbf{H}| = |\mathbf{K}|$.

Now, suppose for the sake of contradiction that $\mathbf{H}$ is not isomorphic to $\mathbf{K}$. Since they are of the same finite order, it must be that one of the cyclic groups comprising $\mathbf{H}$ appears with different multiplicity than in $\mathbf{K}$ (note one of these multiplicites may be 0). Let the order of this cyclic factor be $p^k$ for some prime $p$. We see that $\mathbf{H}$ and $\mathbf{K}$ necessarily have differing numbers of elements of order $p^k$. This implies that $\mathbf{G} \times \mathbf{H}$ and $\mathbf{G} \times \mathbf{K}$ do not have the same number of elements of order $p^k$, and so are not isomorphic, which is a contradiction. Therefore, it must be that $\mathbf{H} \cong \mathbf{K}$. □

Problem 18

Prove that every group of order 35 is cyclic.

*Proof.* Observe first that $35 = 5 \cdot 7$. Consider the Sylow $p$-subgroups of $G$. Sylow's theorem gives

- $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 7$, so $n_5 = 1$
- $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 5$, so $n_7 = 1$

Hence, $\mathbf{G}$ has a unique Sylow 5-subgroup $\mathbf{N}_5$ and a unique Sylow 7-subgroup $\mathbf{N}_7$. The uniqueness of each of these groups implies that they are normal in $\mathbf{G}$.

Observe next that $\mathbf{N}_5 \cap \mathbf{N}_7$ is trivial, since only the identity element can have order dividing both $|\mathbf{N}_5|$ and $|\mathbf{N}_7|$. By the Third Isomorphism theorem, we have

$$\mathbf{N}_5\mathbf{N}_7/\mathbf{N}_7 \cong \mathbf{N}_5/\mathbf{N}_5 \cap \mathbf{N}_7 = \mathbf{N}_5$$

Applying Lagrange's theorem, we have

$$\begin{aligned} \left|\mathbf{N}_5\mathbf{N}_7/\mathbf{N}_7\right| &= |\mathbf{N}_5| \\ \frac{|\mathbf{N}_5\mathbf{N}_7|}{|\mathbf{N}_7|} &= |\mathbf{N}_5| \\ |\mathbf{N}_5\mathbf{N}_7| &= |\mathbf{N}_5||\mathbf{N}_7| = |\mathbf{G}| \end{aligned}$$

Hence, $\mathbf{G} \cong \mathbf{N}_5 \times \mathbf{N}_7$.

Now, both of $\mathbf{N}_5$ and $\mathbf{N}_7$ are cyclic (and so Abelian), since they are of prime order. We claim that $\mathbf{G}$ is also Abelian. Let $(a_1, b_1)$ and $(a_2, b_2)$ be elements of $\mathbf{G}$. It follows that

$$\begin{aligned} (a_1, b_1) * (a_2, b_2) &= (a_1a_2, b_1b_2) \\ &= (a_2a_1, b_2b_1) \\ &= (a_2, b_2)(a_1, b_1) \end{aligned}$$

Hence, $\mathbf{G}$ is Abelian. We see that $\mathbf{G}$ is a finite Abelian group with $k^2$ not dividing its order for all $k > 1$. By problem 20, we conclude that $\mathbf{G}$ is cyclic. □

Problem 19

Describe, up to isomorphism, all groups of order 1225.

*Proof.* Observe first that $1225 = 5^2 7^2$. Consider the Sylow $p$-subgroups of $G$. Sylow's theorem gives

- $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 49$, so $n_5 = 1$
- $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 25$, so $n_7 = 1$

Hence, $\mathbf{G}$ has a unique Sylow 5-subgroup $\mathbf{N}_5$ and a unique Sylow 7-subgroup $\mathbf{N}_7$. The uniqueness of each of these groups implies that they are normal in $\mathbf{G}$.

Observe next that $\mathbf{N}_5 \cap \mathbf{N}_7$ is trivial, since only the identity element can have order dividing both $|\mathbf{N}_5|$ and $|\mathbf{N}_7|$. By the Third Isomorphism theorem, we have

$$\mathbf{N}_5\mathbf{N}_7 / \mathbf{N}_7 \cong \mathbf{N}_5 / \mathbf{N}_5 \cap \mathbf{N}_7 = \mathbf{N}_5$$

Applying Lagrange's theorem, we have

$$\begin{aligned} \left| \mathbf{N}_5\mathbf{N}_7 / \mathbf{N}_7 \right| &= |\mathbf{N}_5| \\ \frac{|\mathbf{N}_5\mathbf{N}_7|}{|\mathbf{N}_7|} &= |\mathbf{N}_5| \\ |\mathbf{N}_5\mathbf{N}_7| &= |\mathbf{N}_5||\mathbf{N}_7| = |\mathbf{G}| \end{aligned}$$

Hence, $\mathbf{G} \cong \mathbf{N}_5 \times \mathbf{N}_7$.

We proceed by showing $\mathbf{N}_5$ is Abelian. Since $\mathbf{N}_5$ is of prime power order, it has a nontrivial center $Z(\mathbf{N}_5)$. Furthermore, $Z(\mathbf{N}_5)$ is normal in $\mathbf{N}_5$, so $\mathbf{N}_5 / Z(\mathbf{N}_5)$ is a group of size 1 or 5. If it is of size 5, then it is cyclic. We claim that this is impossible in general.

**Claim 1.** *If a group $\boldsymbol{G}$ properly contains its center, then $\boldsymbol{G} / Z(\boldsymbol{G})$ is not cyclic.*

*Proof.* Suppose, to the contrary, that $\mathbf{G} / Z(\mathbf{G})$ is cyclic generated by $aZ(\mathbf{G})$. We argue that $G$ is Abelian. Let $b$ and $c$ be elements of $G$. We can find integers $m$ and $n$ so that

$$\begin{aligned} bZ(\mathbf{G}) &= a^m Z(\mathbf{G}) \\ cZ(\mathbf{G}) &= a^n Z(\mathbf{G}) \end{aligned}$$

This further implies that we can find elements $d$ and $e$ in $Z(\mathbf{G})$ so that

$$\begin{aligned} b &= a^m d \\ c &= a^n e \end{aligned}$$

Observe that $d$ and $e$ commute freely with any element since they are in the center. Furthermore, powers of $a$ commute with each other. It follows that

$$\begin{aligned} bc &= (a^m d)(a^n e) \\ &= (a^n e)(a^m d) \\ &= cb \end{aligned}$$

Hence, $\mathbf{G}$ is Abelian, so $Z(\mathbf{G}) = \mathbf{G}$, which contradicts our assumption that $\mathbf{G}$ properly contains its center. Therefore, we conclude that $\mathbf{G} / Z(\mathbf{G})$ is not cyclic. □

Citing the claim above, we conclude that $\mathbf{N}_5 / Z(\mathbf{N}_5)$ is of size 1. In other words, $\mathbf{N}_5$ is Abelian.

Similarly, we can show that $\mathbf{N}_7$ is Abelian (replace every occurrence of "5" with "7" in the argument for $\mathbf{N}_5$).

Applying the Fundamental Theorem of Finite Abelian Groups, we conclude that $\mathbf{G}$ is isomorphic to one of

$$\begin{gathered}\mathbb{Z}_{49} \times \mathbb{Z}_{25} \\ \mathbb{Z}_{49} \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_{25} \times \mathbb{Z}_7 \times \mathbb{Z}_7 \\ \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_5\end{gathered}$$

□

Problem 20

Let $\mathbf{G}$ be a finite Abelian group. Prove that if $|\mathbf{G}|$ is not divisible by $k^2$ for any $k > 1$, then $\mathbf{G}$ is cyclic.

*Proof.* By the Fundamental Theorem of Finite Abelian Groups, $\mathbf{G}$ has a unique decomposition as a direct product of cyclic groups of prime power order. More precisely,

$$\mathbf{G} \cong \mathbb{Z}_{{p_1}^{n_1}} \times \cdots \times \mathbb{Z}_{{p_k}^{n_k}}$$

where each $p_i$ is a prime number and each $n_i \geq 1$. Since $k^2$ does not divide the order of $\mathbf{G}$ for any $k > 1$, it must be that $n_i = 1$ for all $i$ (otherwise, $p_i^2$ divides the order of $\mathbf{G}$ for some $i$). We have now

$$\mathbf{G} \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$$

We see that $|\mathbf{G}| = p_1 \cdots p_k$. We claim that $\mathbf{G}$ has an element of order $p_1 \cdots p_k$, and hence is cyclic.

Consider the element $\underbrace{(1, \ldots, 1)}_{\text{k times}}$ of $\mathbf{G}$. We see that $(1, \cdots, 1)^m = (0, \cdots, 0)$ if and only if $m$ is the least common multiple of $p_1, \cdots, p_k$. Since the $p_i$ are prime (and so relatively prime), $m = p_1 \cdots p_k$. That is, $(1, \cdots, 1)$ has order $p_1 \cdots p_k$, and so $\mathbf{G}$ is cyclic. □