Math 731 Homework 5 (Correction 1)

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1 Extra Problem

Proposition 1.1. *A nonempty collection of subsets of X is a subbase for some filter on X if and only if it has the finite intersection property.*

Proof. ($\Rightarrow$) We establish the contrapositive. That is, suppose $\mathcal{A}$ is a nonempty collection of subsets of X that fails to satisfy the finite intersection property. It follows that there are sets $A_1, \ldots, A_k \in \mathcal{A}$ such that $\bigcap_{i=1}^k A_i = \emptyset$. Consider now the collection $\mathcal{C} = \{\bigcap_{i=1}^n A_i \mid A_i \in \mathcal{A} \text{ for all } i\}$. Evidently, $\emptyset \in \mathcal{C}$, and so it cannot be a base for any filter on X (as no filter contains the empty set). Hence, $\mathcal{A}$ is not a subbase for any filter on X, thus establishing the contrapositive.

($\Leftarrow$) Let $\mathcal{S}$ be a nonempty collection of subsets of X having the finite intersection property. Define $\mathcal{B}$ to be the collection

$$\{F \subset X \mid \bigcap \mathcal{S}' \subset F \text{ for some finite subcollection } \mathcal{S}' \text{ of } \mathcal{S}\}.$$

Observe first that $\mathcal{B}$ is a nonempty collection of nonempty subsets of X, as $\mathcal{S}$ satisfies the finite intersection property. We further claim that $\mathcal{B}$ is in fact a base for a filter on X, and so $\mathcal{S}$ will be a subbase for this filter. To that end, let $B_1, B_2 \in \mathcal{B}$. By definition of $\mathcal{B}$,

$$\begin{aligned} B_1 \cap B_2 &= \left(\bigcap \mathcal{S}_1\right) \cap \left(\bigcap \mathcal{S}_2\right) \qquad \text{(some } \mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}) \\ &= \bigcap (\mathcal{S}_1 \cup \mathcal{S}_2). \end{aligned}$$

Evidently, $\mathcal{S}_1 \cup \mathcal{S}_2$ is a finite subcollection of $\mathcal{S}$, as both $\mathcal{S}_1$ and $\mathcal{S}_2$ are. Hence, $B_1 \cap B_2 \in \mathcal{B}$, and so $\mathcal{B}$ is a base for the filter $\{F \subset X \mid B \subset F \text{ for some } B \in \mathcal{B}\}$. $\square$

2 Problem 12B1

Proposition 2.1. *The intersection of any number of filters on X is a filter on X.*

Proof. Let $\mathcal{F}$ be the intersection of filters $\mathcal{F}_\alpha$ for $\alpha \in \Gamma$. We claim that $\mathcal{F}$ is itself a filter.

First, observe that $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$, since this is the case for each of the $\mathcal{F}_\alpha$.

Next, let F_1 and F_2 be elements of $\mathcal{F}$. It follows that F_1 and F_2 belong to $\mathcal{F}_\alpha$ for all $\alpha \in \Gamma$, and so $F_1 \cap F_2 \in \mathcal{F}_\alpha$ for all $\alpha \in \Gamma$. Hence, $F_1 \cap F_2 \in \mathcal{F}$.

Finally, let $F \in \mathcal{F}$ and let G be any set containing F. Since $F \in \mathcal{F}_\alpha$ for all $\alpha \in \Gamma$, it follows that $G \in \mathcal{F}_\alpha$ for all $\alpha \in \Gamma$. Hence, $G \in \mathcal{F}$.

Taken together, the above three observations verify that $\mathcal{F}$ is indeed a filter, as desired. □

3 Problem 12B3

Proposition 3.1. *Every filter is the intersection of the ultrafilters containing it.*

Proof. Let $\mathcal{F}$ be a filter and let $\mathcal{U} = \bigcap_{\alpha \in \Gamma} \mathcal{U}_\alpha$, where the $\mathcal{U}_\alpha$ constitute all ultrafilters containing $\mathcal{F}$. Certainly, $\mathcal{F} \subset \mathcal{U}$. Suppose, for the purpose of contradiction, that this inclusion is strict. It follows that there is some set A with $A \notin \mathcal{F}$ and $A \in \mathcal{U}$. Consider the filter $\mathcal{G}$ generated by $\mathcal{F} \cup \{A^c\}$ (this is indeed a filter, since $A \notin \mathcal{F}$ and so $\emptyset \notin \mathcal{G}$). We have that $\mathcal{F} \subset \mathcal{G} \subset \mathcal{U}_\beta$, for some $\beta \in \Gamma$. Now, since $A \in \mathcal{U}$, $A \in \mathcal{U}_\beta$, but also $A^c \in \mathcal{U}_\beta$, which is a contradiction. Therefore, it must be that $\mathcal{F} = \mathcal{U}$. □