Math 730 Homework 5 (Correction 1)

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1 Extra Problem 1

Lemma 1.1. *Let* $\hat{}$ *be a Čech closure operation. If* $A \subset B$*, then* $\hat{A} \subset \hat{B}$*.*

Proof. It follows directly from the properties of Čech closure operations that

$$\begin{aligned} A \subset B \quad &\Rightarrow \quad A \cup B = B \\ &\Rightarrow \quad \widehat{A \cup B} = \hat{B} \\ &\Rightarrow \quad \hat{A} \cup \hat{B} = \hat{B}. \end{aligned}$$

Now, $\hat{A} \subset \hat{A} \cup \hat{B} = \hat{B}$, and so $\hat{A} \subset \hat{B}$, as desired. □

Proposition 1.2. *If we define a set in a Čech closure space* $(X, \hat{})$ *to be closed if* $A = \hat{A}$*, then the result is a topology.*

Proof. Let $\mathcal{F}$ denote the collection of subsets of X that are closed with respect to $\hat{}$. By Homework 4, Problem 2, if $\mathcal{F}$ satisfies

(F-a) the intersection of an arbitrary collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$,

(F-b) the union of a finite collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$, and

(F-c) the sets $\emptyset$ and X belong to $\mathcal{F}$,

then $\tau = \{F^c \mid F \in \mathcal{F}\}$ is a topology on X. We verify that $\mathcal{F}$ indeed satisfies each of these three properties.

Claim. *The intersection of an arbitrary collection of elements of* $\mathcal{F}$ *belongs to* $\mathcal{F}$*.*

Proof. Let $F_\alpha \in \mathcal{F}$ for all α belonging to some indexing set I. As $\hat{}$ is a Čech closure operation,

$$\bigcap_{\alpha \in I} F_\alpha \subset \widehat{\bigcap_{\alpha \in I} F_\alpha}.$$

To see the reverse inclusion, observe that

$$\begin{aligned} & \bigcap_{\alpha \in I} F_\alpha \subset F_\alpha \text{ for each } \alpha \in I \\ \Rightarrow \quad & \widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \hat{F}_\alpha \text{ for each } \alpha \in I \quad \text{(by 1.1)} \\ \Rightarrow \quad & \widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \bigcap_{\alpha \in I} \hat{F}_\alpha \\ \Rightarrow \quad & \widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \bigcap_{\alpha \in I} F_\alpha \qquad \text{(as each } F_\alpha \text{ is closed).} \end{aligned}$$

Hence, $\bigcap_{\alpha \in I} F_\alpha = \widehat{\bigcap_{\alpha \in I} F_\alpha}$ (i.e. $\bigcap_{\alpha \in I} F_\alpha$ is closed in $\mathcal{F}$). □

Claim. *The union of a finite collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.*

Proof. Let $F_i \in \mathcal{F}$ for $1 \leq i \leq n$. We have that

$$\begin{aligned} \bigcup_{i=1}^{n} F_i &= \bigcup_{i=1}^{n} \hat{F}_i \qquad \text{(since } F_i = \hat{F}_i \text{ for all } i\text{)} \\ &= \widehat{\bigcup_{i=1}^{n} F_i} \qquad \text{(by induction on } i\text{, using the fact that } \widehat{A \cup B} = \hat{A} \cup \hat{B} \text{ for the Čech closure operation } \hat{\ }\text{)} \end{aligned}$$

□

Claim. *The sets $\emptyset$ and X belong to $\mathcal{F}$.*

Proof. We have that $\emptyset = \hat{\emptyset}$ by definition of Čech closure operation, so $\emptyset \in \mathcal{F}$. Now,

$$\begin{aligned} X &\subset \hat{X} \qquad \text{(since } \hat{\ } \text{ is a Čech closure operation)} \\ &\subset X \qquad \text{(since } \hat{\ } : \mathcal{P}(X) \to \mathcal{P}(X)\text{).} \end{aligned}$$

Hence, $X = \hat{X}$, and so $X \in \mathcal{F}$. □

Therefore, $\tau = \{F^c \mid F \in \mathcal{F}\}$ is a topology on X, as desired. □

2 Problem 3A1

Proposition 2.1. *If $\mathcal{F}$ is the collection of all closed, bounded subsets of $\mathbb{R}$ (in its usual topology), together with $\mathbb{R}$ itself, then $\mathcal{F}$ is the family of closed sets for a topology on $\mathbb{R}$ strictly weaker than the usual topology.*

Proof. We proceed by verifying properties F-a, F-b, and F-c on the family of sets $\mathcal{F}$.

Claim. *The intersection of an arbitrary collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.*

Proof. Let $F_\alpha \in \mathcal{F}$ for all α belonging to some indexing set I. In $\mathbb{R}$, the intersection of an arbitrary collection of closed sets is closed. Hence, $\bigcap_{\alpha \in I} F_\alpha$ is closed. Furthermore, each F_α is bounded, and so $\bigcap_{\alpha \in I} F_\alpha$ (which is contained in F_α for every α) is also bounded. Thus, $\bigcap_{\alpha \in I} F_\alpha$ is closed and bounded, and so belongs to $\mathcal{F}$. □

Claim. *The union of a finite collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.*

Proof. Let $F_i \in F$ for all $1 \leq in$. In $\mathbb{R}$, the union of a finite collection of closed sets is closed. Hence, $\bigcup_{i=1}^{n} F_i$ is closed. Now, each F_i is bounded by some $B(0, \epsilon_i)$. Take $\epsilon = \max\{\epsilon_i \mid 1 \leq i \leq n\}$. We see that $F_i \subset B(0, \epsilon)$ for all $1 \leq i \leq n$, and so $\bigcup_{i=1}^{n} F_i \subset B(0, \epsilon)$. Thus, $\bigcup_{i=1}^{n} F_i$ is closed and bounded, and so belongs to $\mathcal{F}$. □

Claim. *The sets $\emptyset$ and $\mathbb{R}$ belong to $\mathcal{F}$.*

Proof. We have that $\mathbb{R} \in \mathcal{F}$ by definition. Now, $\emptyset$ is trivially closed and $\emptyset \subset B(0, 1)$. Hence, $\emptyset$ is closed and bounded, and so belongs to $\mathcal{F}$. □

Therefore, $\mathcal{F}$ is the family of closed subsets for the topology $\tau = \{F^c \mid F \in \mathcal{F}\}$. To see that τ is weaker than the usual Euclidean topology (call it τ'), observe that $B(0, 1) \in \tau'$ (since open balls are open), but $B(0, 1) \notin \tau$ (since $B(0, 1)^c = \mathbb{R} \setminus B(0, 1)$ is unbounded). □

3 Problem 3E2

Proposition 3.1. *Let X be a metrizable space whose topology is generated by a metric ρ. The closure of a set $E \subset X$ is given by*

$$\overline{E} = \{y \in X \mid \rho(E, y) = 0\}.$$

Proof. ($\subset$) Let $x \in \overline{E}$. It follows that,

$$\begin{aligned} x \in \overline{E} &\Rightarrow G \cap E \neq \emptyset \text{ for all open } G \text{ containing } x \\ &\Rightarrow B(x, \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0 \\ &\Rightarrow d(E, x) < \epsilon \text{ for all } \epsilon > 0 \\ &\Rightarrow d(E, x) = 0 \\ &\Rightarrow x \in \{y \in X \mid \rho(E, y) = 0\}. \end{aligned}$$

Hence, $\overline{E} \subset \{y \in X \mid \rho(E, y) = 0\}$.

($\supset$) Let $x \in \{y \in X \mid \rho(E, y) = 0\}$. It follows that,

$$\begin{aligned} x \in \{y \in X \mid \rho(E, y) = 0\} &\Rightarrow \rho(E, x) = 0 \\ &\Rightarrow B(x, \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0. \end{aligned}$$

Now, for any open G containing x, there must exist $\epsilon_0 > 0$ such that $B(x, \epsilon_0) \subset G$ (by definition of openness in a metrizable space). Hence,

$$\begin{aligned} B(x, \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0 &\Rightarrow G \cap E \neq \emptyset \text{ for all open } G \text{ containing } x \\ &\Rightarrow x \in \overline{E}. \end{aligned}$$

Hence, $\{y \in X \mid \rho(E, y) = 0\} \subset \overline{E}$. □

4 Problem 3E3

Proposition 4.1. *Let X be a metrizable space whose topology is generated by a metric ρ. The closed disk $U(x, \overline{\epsilon}) = \{y \in X \mid \rho(x, y) \leq \epsilon\}$ is closed in X, but may not be the closure of the open disk $U(x, \epsilon)$.*

Proof. We show, equivalently, that $U(x, \overline{\epsilon})^c$ is open. As X is a metrizable space, we need to establish that, for any $z \in U(x, \overline{\epsilon})^c$, there exists $\epsilon' > 0$ such that $B(z, \epsilon') \subset U(x, \overline{\epsilon})^c$. To that end, let $\epsilon' = \rho(x, z) - \epsilon$ (which, a priori, may or may not be positive). Now,

$$\begin{aligned} B(z, \epsilon') \subset U(x, \overline{\epsilon})^c &\Leftrightarrow B(z, \epsilon') \subset \{y \in X \mid \rho(x, y) > \epsilon\} \\ &\Leftrightarrow \rho(x, w) > \epsilon \text{ for all } w \in B(z, \epsilon'). \end{aligned}$$

Applying the triangle inequality, we have

$$\begin{aligned} \rho(x, z) &\leq \rho(x, w) + \rho(w, z) \\ &< \rho(x, w) + (\rho(x, z) - \epsilon) && \text{(note now that } \rho(x, z) > \epsilon \text{, so } \rho(x, z) - \epsilon > 0). \end{aligned}$$

Rearranging the terms yeilds that $\rho(x, w) > \epsilon$, as desired.

As an example when $U(x, \overline{\epsilon}) \neq \overline{U(x, \epsilon)}$, let $X = \mathbb{R}$ and ρ be the discrete metric. Observe that $U(0, \overline{1}) = \mathbb{R}$, but

$$\begin{aligned} \overline{U(0,1)} &= \bigcap \{K \mid K \text{ is closed and contains } U(0,1)\} \\ &= \bigcap \{K \mid K \text{ is closed and contains } \{0\}\} \\ &= \{0\} && \text{(since } \{0\} \text{ itself is closed in a metric space).} \end{aligned}$$

□