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Math 704
Homework 4

Problem 1

Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ with $|f_n(x)| < \infty$ for almost every x . Show that there exists a sequence c_n of positive real numbers such that

$$\frac{f_n(x)}{c_n} \rightarrow 0 \text{ for almost every } x.$$

Proof. For a fixed value of n , define the set F_k to be $\{x \in [0, 1] : |f_n(x)| \geq k\}$ (which are measurable, since f_n measurable). We see that $F_1 \supseteq F_2 \supseteq \dots$ (as k grows larger, F_k can only get smaller). Now, $m(F_1) \leq 1 < \infty$, and so $\lim_{k \rightarrow \infty} m(F_k) = m(\bigcap_{k=0}^{\infty} F_k)$. Since $|f_n(x)| < \infty$ for almost every x , it must be that $m(\bigcap_{k=0}^{\infty} F_k) = 0$, which implies that there is some k_n such that $m(F_{k_n}) < \frac{1}{2^n}$. In other words,

$$m(\{x \in [0, 1] : |f_n(x)| \geq k_n\}) < \frac{1}{2^n}$$

or, equivalently

$$m(\{x \in [0, 1] : \left| \frac{f_n(x)}{nk_n} \right| \geq \frac{1}{n}\}) < \frac{1}{2^n}$$

Now, for each n , define E_n to be the set $\{x \in [0, 1] : \left| \frac{f_n(x)}{nk_n} \right| \geq \frac{1}{n}\}$. We have now a countable family $\{E_n\}$ of measurable subsets of \mathbb{R} with $\sum_{k=1}^{\infty} m(E_k) < \infty$. Define E to be the set $\{x \in \mathbb{R} : x \in E_k \text{ for infinitely many } k\}$. The Borel-Cantelli lemma implies $m(E) = 0$. In other words, the subset of $[0, 1]$ whose image is nonzero under $\left| \frac{f_n(x)}{nk_n} \right|$ for infinitely-many n has measure zero. Therefore,

$$\frac{f_n(x)}{nk^n} \rightarrow 0 \text{ for almost every } x$$

□

Problem 2

Let $\chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Show that there is no everywhere continuous function f on \mathbb{R} such that

$$f(x) = \chi_{[0,1]}(x) \text{ almost everywhere.}$$

Proof. Suppose $f(x) = \chi_{[0,1]}(x)$ almost everywhere. Consider the set $(-\frac{\delta}{2}, 0) \cup (0, \frac{\delta}{2})$ for any $\delta > 0$. Since $f(x) = \chi_{[0,1]}(x)$ almost everywhere, we can find $x_0 \in (-\frac{\delta}{2}, 0)$ such that $f(x_0) = 0$ and $x_1 \in (0, \frac{\delta}{2})$ such that $f(x_1) = 1$. Now, let $0 < \epsilon < 1$ be given. Regardless of how small we make δ , we have

$$|x_0 - x_1| < \delta \text{ yet } |f(x_0) - f(x_1)| = |0 - 1| = 1 > \epsilon$$

Therefore, f is not continuous everywhere. □

Problem 3

Let \mathcal{N} denote the measurable set constructed in the text. Recall that measurable subsets of \mathcal{N} have measure zero. Show that the set $\mathcal{N}^c = [0, 1] \setminus \mathcal{N}$ satisfies $m^*(\mathcal{N}^c) = 1$, and conclude that

$$m^*(\mathcal{N}) + m^*(\mathcal{N}^c) \neq m^*(\mathcal{N} \cup \mathcal{N}^c)$$

although \mathcal{N} and \mathcal{N}^c are disjoint.

Proof. Suppose, to the contrary, that $0 \leq m^*(\mathcal{N}^c) < 1$. We can find a measurable set U such that $N^c \subset U \subset [0, 1]$ with $m(U) = 1 - \epsilon$ for some small ϵ . Now, $U^c \subseteq \mathcal{N}$, and so $m(U^c) = 0$ (since measurable subsets of a non-measurable set have measure zero). We have that

$$\begin{aligned} 1 - \epsilon &= m(U) + m(U^c) \\ &= m(U \cup U^c) \\ &= m([0, 1]) \\ &= 1 \end{aligned}$$

which is a contradiction. Hence, $m^*(\mathcal{N}^c) = 1$. Now, during the course of the proof that \mathcal{N} is non-measurable, it was shown that $m^*(\mathcal{N}) > 0$. So, we have $m^*(\mathcal{N}) + m^*(\mathcal{N}^c) > 1$. On the other hand, we have that $m^*(\mathcal{N} \cup \mathcal{N}^c) = m^*([0, 1]) = 1$. Therefore, $m^*(\mathcal{N}) + m^*(\mathcal{N}^c) \neq m^*(\mathcal{N} \cup \mathcal{N}^c)$. □

Problem 4

Give an example of a measurable function f and a continuous function Φ so that $f \circ \Phi$ is non-measurable.

(Hint: Let F be as in problem 2d and define $G : [0, 1] \rightarrow [0, 2]$ by $G(x) = x + F(x)$. Check that G is strictly increasing, continuous and onto. Take C_2 equal to the standard Cantor set and $C_1 = G(C_2)$. Show $m(C_1) = 1$. Now let $\Phi = G^{-1}$. Let $N \subset C_1$ be non-measurable, and take $f = \chi_{\Phi(N)}$.)

Use the construction in the hint to show that there exists a Lebesgue measurable set that is not a Borel set.

Proof. Recall that

$$F(x) = \begin{cases} \text{the ternary expansion of } x & \text{if } x \in C_2 \\ F(y) \text{ where } y \text{ is the greatest element of } C_2 \text{ such that } y < x & \text{if } x \notin C_2 \end{cases}$$

We see that $F(x)$ is nondecreasing. Since x is strictly increasing, we have that $G(x) = x + F(x)$ is strictly increasing. We have shown that $F(x)$ is continuous, so $G(x)$ is continuous (since the sum of two continuous functions is continuous). Next, observe that $G(0) = 0$ and $G(1) = 2$. By the Intermediate Value Theorem, we have that G is onto $[0, 2]$.

Recall that C_2^c is comprised of open intervals (a, b) where $F(a) = F(b)$. Let $(a, b) \in C_2^c$ have measure l . We have

$$\begin{aligned} G(a) &= a + F(a) \\ G(b) &= b + F(b) \end{aligned}$$

Since $F(a) = F(b)$, we see that $G(b) - G(a) = a - b = l$. Furthermore, since G is strictly increasing and continuous, it must be that the open interval $(G(a), G(b))$ has measure l . Repeating this argument for all open subsets of C_2^c , we see that $m(G(C_2^c)) = m(C_2^c) = 1$. Since G is onto $[0, 2]$, we have $m(C_1) = m(G(C_2)) = m([0, 2] \setminus G(C_2^c)) = 1$.

Since G is continuous and onto $[0, 2]$, G^{-1} is continuous, so we will take Φ in the problem statement to be G^{-1} . Let N denote some non-measurable subset of C_1 and consider the function $\chi_{G^{-1}(N)}$. We see that $\chi_{G^{-1}(N)}$ is measurable since

$$\{\chi_{G^{-1}(N)} > a\} = \begin{cases} [0, 2] & \text{if } -\infty \leq a \leq 0 \\ \{x : G(x) \in N\} & \text{if } 0 < a \leq 1 \\ \emptyset & \text{if } a > 1 \end{cases}$$

The first and third cases are obviously measurable. To see that the second case is measurable, observe that $\{x : G(x) \in N\} \subseteq C_2$, which has measure zero. Subsets of sets with measure zero are measurable (also with measure zero), and so it must be that $\{x : G(x) \in N\}$ is measurable. So, we will take f in the problem statement to be $\chi_{G^{-1}(N)}$.

We show next that $\chi_{G^{-1}(N)} \circ G^{-1}$ is non-measurable. Consider the set $\{\chi_{G^{-1}(N)} \circ G^{-1} > 0\}$. Since $\chi_{G^{-1}(N)} \circ G^{-1}$ outputs only 0 or 1, this set is equivalent to $\{\chi_{G^{-1}(N)} \circ G^{-1} = 1\}$. Translating the notation, we see that this set is all of N , which is non-measurable by definition. Hence, $\chi_{G^{-1}(N)} \circ G^{-1}$ is non-measurable.

Consider the set $G^{-1}(N)$. Since $G^{-1}(N) \subseteq C_2$ and C_2 has measure zero, it must be that $G^{-1}(N)$ is Lebesgue measurable with measure zero. We might proceed by demonstrating that $G^{-1}(N)$ cannot be obtained by from open sets of \mathbb{R} using the properties of σ -algebras. (I cannot determine how to go about this.) \square