Math 731 Homework 4 (Correction 1)

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1 Problem 16D1

Proposition 1.1. *The Sorgenfrey line* $\boldsymbol{E}$ *is Lindelöf.*

Proof. Let $\mathcal{C}$ be a basic open (in $\mathbf{E}$) cover of $\mathbb{R}$. That is, $\mathcal{C} = \{[x_\alpha, y_\alpha) \mid x_\alpha, y_\alpha \in \mathbb{R}, \alpha \in \Gamma\}$. Let $\mathcal{C}' = \{(x_\alpha, y_\alpha) \mid [x_\alpha, y_\alpha) \in \mathcal{C}\}$. Finally, let $A = \bigcup C'$. Evidently, $\mathcal{C}'$ covers A. Now, observe that $\mathcal{C}'$ contains open sets from the usual topology on $\mathbb{R}$. Since $\mathbb{R}$ with the usual topology is second-countable, it is Lindelöf. Hence, there is a countable subcollection of $\mathcal{C}'$ (and so of $\mathcal{C}$) covering A.

Next, we claim that A misses only countably-many points of $\mathbb{R}$. To see this, let x and y be distinct real numbers that are not elements of A. Without loss of generality, let $x < y$. Since $\mathcal{C}$ covers $\mathbb{R}$, it follows that x and y are left endponts of distinct intervals of $\mathcal{C}$. Denote these intervals by $[x, x')$ and $[y, y')$, respectively. Evidently, $[x, x')$ and $[y, y')$ are disjoint (if not, then $y \in (x, x')$, and so $y \in A$). Hence, we can identify with each of x and y a distinct rational number belonging to (x, x') and (y, y'), respectively. Thus, A misses only countably-many elements of $\mathbb{R}$, and so there is a countable subcollection of $\mathcal{C}$ covering A.

We have shown that $\mathcal{C}$ admits a countable subcover for both A and $\mathbb{R} \setminus A$. Taking these subcovers together yields the desired subcover of $\mathbb{R}$. $\square$

Corollary 1.2. *The Sorgenfrey line is a* T_4*-space.*

Proof. Since the Soregenfrey line is regular and Lindelöf, it is normal. Furthermore, it is T_1. Taken together, we have that the Sorgenfrey line is T_4. $\square$

2 Problem 16D2

Proposition 2.1. *Every uncountable subset of a Lindelöf space contains an accumulation point.*

Proof. We establish the contrapositive. To that end, let A be a subset of a Lindelöf space X such that A contains no accumulation points. It follows that, for each $x \in X$, there is an open neighborhood U_x of x with $|U_x \cap A|$ finite. Let $\mathcal{C}$ denote the collection $\{U_x \mid x \in X\}$. Since $\mathcal{C}$ is an open cover of the Lindelöf space X, $\mathcal{C}$ admits a countable subcover $\mathcal{C}'$ of X (and so of A). Since $|A \cap U|$ is finite for all U belonging to the countable collection $\mathcal{C}'$, it follows that A is countable, thus establishing the contrapositive. $\square$