Problem Set 3

MATH 778C, Spring 2009, Austin Mohr (with John Boozer)

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1. Show directly that $P_1(s) \Rightarrow P_1(t)$ for all $t \leq s$.

Proof. Given G, let H_s be a subgraph of G on s vertices such that $\#(H_s \sqsubset G) = n^s 2^{-\binom{s}{2}}(1 + o(1))$. Let H_{s-1} be a graph on $s-1$ vertices. Each embedding of H_{s-1} can be extended to n embeddings of s-vertex graphs by adding one of the n vertices as the s^{th} vertex. We count these extended embeddings and divide by n.

Each embedding is a copy of our original graph plus an additional vertex. There are 2^{s-1} such graphs, each embedded $n^s 2^{-\binom{s}{2}}(1 + o(1))$ times. So

$$\begin{aligned}
\#(H_{s-1} \sqsubset G) &= \frac{2^{s-1}}{n} \cdot \#(H_s \sqsubset G) \\
&= \frac{2^{s-1}}{n}(n^s 2^{-\binom{s}{2}}(1 + o(1))) \\
&= n^{s-1} 2^{-\binom{s-1}{2}}(1 + o(1)).
\end{aligned}$$

□

2. Show that, for a sequence of G's with $n \to \infty$,

$$P_1(3) \not\Rightarrow P_1(4).$$

Proof. Let G_n be comprised of two K_n and one $K_{n,n}$ connected in the following way:

- there are no edges between the K_n
- between the $K_{n,n}$ and each K_n, every possible edge occurs with probability $\frac{1}{2}$

To see that G_n satisfies $P_1(3)$, observe first that it suffices to count only copies of K_3 and P^1 (here, we use P^j to denote a path of length j to distinguish from the property names), since G_n is self-complementary. Hence, if the number of K_3 is correct, then the number of $\overline{K_3}$ is correct. Similarly, if the number of P^1 is correct, then the number of $\overline{P^1} = P^2$ is correct.

We have the following cases for the K_3.

If all three vertices belong to a K_n, then there are $2 \cdot n \cdot n \cdot n = 2n^3$ copies of K_3 (2 choices for which K_n the vertices belong to and n choices for each vertex).

If exactly two vertices belong to a K_n, then there are $2 \cdot 3 \cdot n \cdot n \cdot 2n \cdot \frac{1}{4} = 3n^3$ copies of K_3 (2 choices for which K_n the two vertices belong to, 3 choices for which vertex of the K_3 belongs to the $K_{n,n}$, n possible choices for each of the two vertices within the K_n, and $2n$ choices for the vertex in the $K_{n,n}$ having a $\frac{1}{4}$ probability of successfully linking up with the vertices in the K_n).

If exactly one vertex belongs to a K_n, then there are $2 \cdot 3 \cdot n \cdot n \cdot n \cdot 2 \cdot \frac{1}{4} = 3n^3$ (2 choices for which K_n the two vertices belong to, 3 choices for which vertex of the K_3 belongs to the K_n, n possible choices for the vertex within the K_n, n choices for the vertex in the left partite set of the $K_{n,n}$, n choices for the vertex in the right partite set of the $K_{n,n}$, 2 ways to flip the partite sets, and a $\frac{1}{4}$ probability of successfully linking up with the vertex in the K_n).

In total, we determine there are $8n^3$ copies of K_3 in G_n (which has $4n$ vertices), as desired.

Next, we consider the following cases for the P^1. For convenience, we view P^1 as a single edge and an isolated vertex.

If the edge lies in a K_n, there are $2 \cdot n \cdot n \cdot (n + \frac{2n}{4}) = 3n^3$ copies of P^1 (2 choices for which K_n the edge belongs to, n choices for each of the two vertices comprising the edge, and $n + \frac{2n}{4}$ choices for the isolated vertex representing placement in the other K_n or the $\frac{1}{4}$ chance of successful placement in the $K_{n,n}$, respectively).

If the edge lies in the $K_{n,n}$, there are $2 \cdot n \cdot n \cdot n \cdot 2 \cdot \frac{1}{4} = n^3$ copies of P^1 (2 choices for which K_n the isolated vertex belongs to, n choices for the vertex in the left partite set of the $K_{n,n}$, n choices for the vertex in the right partite set of the $K_{n,n}$, 2 ways to flip the partite sets, and a $\frac{1}{4}$ probability of keeping the isolated vertex isolated).

If the edge lies between a K_n and the $K_{n,n}$, there are $2 \cdot n \cdot 2n \cdot 2 \cdot \frac{1}{2} \cdot (\frac{n}{2} + \frac{n}{2}) = 4n^3$ copies of P^1 (2 choices for which K_n the isolated vertex belongs to, n choices for the vertex in the K_n, $2n$ choices for the vertex in the $K_{n,n}$, 2 permutations of the labels of these vertices, a $\frac{1}{2}$ chance of successfully linking these vertices, and finally an $\frac{n}{2}$ chance of placing the isolated vertex in the same partite set of the $K_{n,n}$ or a $\frac{n}{2}$ chance of placing the isolated vertex in the other K_n).

In total, we determine there are $8n^3$ copies of P^1 in G_n (which has $4n$ vertices), as desired.

Next, we demonstrate that there are too many (more than $4n^4$) copies of K_4 in G_n, and so conclude that G_n does not satisfy $P_1(4)$. We again consider cases.

If all four vertices lie in a K_n, there are $2 \cdot n \cdot n \cdot n \cdot n = 2n^4$ copies of K_4 (2 choices for which K_n the vertices belong to, and n choices for each of the vertices within the K_n).

If exactly three of the vertices are in a K_n, there are $2 \cdot n \cdot n \cdot n \cdot 2n \cdot \frac{1}{8} \cdot 4 = 2n^4$ copies of K_4 (2 choices for which K_n the vertices belong to, n choices for each of these three vertices, $2n$ choices for the vertex in the $K_{n,n}$, and a $\frac{1}{8}$ probability of successfully linking all the vertices, and 4 ways to permute the labels).

If exactly two of the vertices are in a K_n, there are $2 \cdot n \cdot n \cdot n \cdot n \cdot \frac{1}{16} \cdot \binom{4}{2} = \frac{3}{4}n^4$ copies of K_4 (2 choices for which K_n the vertices belong to, n choices for each vertex within the K_n, n choices for the vertex in the left partite set of $K_{n,n}$, n choices for the vertex in the right partite set of $K_{n,n}$, a $\frac{1}{16}$ chance of successfully linking the vertices, and $\binom{4}{2}$ ways to permute the labels).

We have already accounted for more than $4n^4$ copies of K_4 in G_n. Hence, G_n does not satisfy $P_1(4)$.

Therefore, $P_1(3) \not\Rightarrow P_1(4)$. $\square$

3. Show that there is a universal constant C so that for any collection of n line segments in the plane, there is some subset S of them with

$$\left| e(S) - \frac{|S|^2}{4} \right| \geq Cn^2.$$

where $e(S)$ means the number of pairs of segments in S which cross one another.

Proof. Define the incidence matrix G where each vertex represents a line segment in the plane and two vertices are adjacent if and only if their corresponding line segments intersect. Observe that the negation of P_4 implies the desired claim. Hence, it suffices to show that G is not quasirandom. We accomplish this by providing a graph that cannot occur as an induced subgraph of G, thereby violating $P_1(s)$.

Let H be a copy of K_5 in which every edge has been subdivided. Consider the "original" vertices of H (that is, those vertices that were present before the subdivisions). None of these vertices are adjacent in H, and so represent line segments in the plane that are pairwise non-intersecting. Call this set of line segments A. Now, the vertices arising as a result of the subdivisions specify another set of line segments (call this set of line segments B). The graph H specifies that each pair of line segments in A is intersected by a single line segment in B in a bijective manner and that all line segments in B are mutually non-intersecting. This is equivalent to drawing K_5 in the plane without intersection, which is impossible. Therefore, H cannot be a subgraph of G, which violates $P_1(15)$, thus asserting the desired claim. $\square$

4. Show that a random graph is quasirandom.

Proof. We proceed by demonstrating that the random graph satisfies P_5. We seek to employ Chebyshev's Inequality,

$$\Pr\left(|X - E[X]| \geq k\sigma\right) \leq \frac{1}{k^2}$$

where our random variable X is the number of common neighbors. On an n-vertex graph, we take $\mu = \frac{n-2}{4}$ as the expected number of common neighbors of some fixed vertices v and w. To determine the total variance, consider the variance at a single vertex u. Now, u can either be a common neighbor of v, w or it cannot, so X can take on only values 0 or 1. Hence, we obtain

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \frac{1}{4} - (\frac{1}{4})^2 \\ &= \frac{3}{16} \end{aligned}$$

Summing over all possible u, we obtain a total variance of $\frac{3}{16}(n-2)$. Hence, $\sigma = c\sqrt{n}$, some constant c. Returning to Chebyshev, we have

$$\begin{aligned} \Pr\left(|X - E[X]| \geq k\sigma\right) &\leq \frac{1}{k^2} \\ \Pr\left(\left|X - \frac{n-2}{4}\right| \geq kc\sqrt{n}\right) &\leq \frac{1}{k^2} \\ \Pr\left(\left|X - \frac{n-2}{4}\right| = o(n)\right) &\geq \frac{k^2-1}{k^2} \end{aligned}$$

for all $k = o(\sqrt{n})$, as desired. □

5. Define the Paley Graph G_p, for p a prime congruent to 1 mod 4, by setting $V(G_p) = \mathbb{Z}_p$ and $\{x, y\} \in G_p$ iff $x-y$ is a quadratic residue (i.e., $x-y = s^2$ for some $s \in \mathbb{Z}_p$). Show that G_p is quasirandom.

Proof. We count the number of common neighbors for any two vertices $v, w \in G_p$. In other words, we count the number of solutions to the equation

$$v - w = s + t,$$

where s and t are quadratic residues. Since $p \equiv 1 \pmod 4$, $-t$ is a quadratic residue whenever t is a quadratic reside. Hence, we can consider the equivalent equation

$$v - w = s - t.$$

Consider,

$$\begin{aligned}
v - w &= (x+h)^2 - x^2 \\
v - w &= 2hx + h^2 \\
v - w - h^2 &= 2hx \\
(2h)^{-1}(v - w - h^2) &= x
\end{aligned}$$

So long as $h \neq 0$, we get one solution for each h (not necessarily distinct).

If a is a solution to

$$(x+h)^2 - x^2 = v - w$$

then $-a$ is a solution to

$$(x-h)^2 - x^2 = v - w,$$

so if we have a, we don't need $-a$. So, we throw out exactly half of our solutions.

If a is a solution to

$$(x+h)^2 - x^2 = v - w$$

then it is also a solution to

$$(x - 2a - h) - x^2 = v - w.$$

So, a occurs at least twice in our list of solutions so long as $h \neq -2a - h$. Now, $h = -2a - h$ implies $a^2 = v - w$, which happens only once. Furthermore, if $a = 0$, then $x - 2a - h = x - h$, and we have already thrown out either the solution to $(x-h)^2 - x^2 = v - w$ or $(x+h)^2 - x^2 = v - w$. So, we throw out nearly half of our solutions. With $p - 1$ possibilities for h, if $v - w$ is not a quadratic residue, we have at most $\frac{p-1}{4}$ distinct solutions. If $v - w$ is a quadratic residue, we have at most $\frac{p+3}{4}$ distinct solutions. This gives P_5, and so G_p is quasirandom. □