Math 731 Homework 3 - Correction 1

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1 The One-Point Compactification: Construction

The procedure used to to obtain the one-point compactification X^* of a locally compact, non-compact Hausdorff space X can be applied to any space Y. That is, $Y^* = Y \cup \{p\}$ with neighborhoods $y \in Y$ unchanged in Y^* while neighborhoods of p have the form $\{p\} \cup (Y - L)$, where L is a subset of Y with compact closure. Y^* is called the *Alexandroff extension* of Y.

2 Problem 19A1

Proposition 2.1. *The assignment of neighborhoods in Y^* described above is valid.*

Proof. Recall that one can define a topology on a set Y by specifying, for each $x \in Y$, a set $\mathcal{U}_x$ (called the *neighborhood system at x*) satisfying

1. If $U \in \mathcal{U}_x$, then $x \in U$.
2. If $U, V \in \mathcal{U}_x$, then $U \cap V \in \mathcal{U}_x$.
3. If $U \in \mathcal{U}_x$, then there is a set $V \in \mathcal{U}_x$ such that $U \in \mathcal{U}_y$ for each $y \in V$.
4. If $U \in \mathcal{U}_x$ and $U \subset V$, then $V \in \mathcal{U}_x$.

Using this approach, we call a set open whenever it contains a neighborhood of each of its points.

Observe that, since Y is a topological space, there already exists a neighborhood system satisfying these requirements for each $x \in Y$. Let $\mathcal{U}_p$ contain all sets of the form $\{p\} \cup (Y - L)$, where L is a subset of Y with compact closure. We proceed by verifying that $\mathcal{U}_p$ satisfies the above requirements.

Claim. *If $U \in \mathcal{U}_p$, then $p \in U$.*

Proof. By definition, $U = \{p\} \cup (Y - L)$ (some L). Hence, $p \in U$. $\square$

Claim. *If $U, V \in \mathcal{U}_p$, then $U \cap V \in \mathcal{U}_p$.*

Proof. Let $U = \{p\} \cup (Y - L)$ and $V = \{p\} \cup (Y - K)$, where L and K are subsets of Y with compact closure. It follows that

$$\begin{aligned} U \cap V &= [\{p\} \cup (Y - L)] \cap [\{p\} \cup (Y - K)] \\ &= \{p\} \cup (Y - (L \cup K)). \end{aligned}$$

Now, $\mathrm{Cl}_Y(L \cup K) = \mathrm{Cl}_Y(L) \cup \mathrm{Cl}_Y(K)$, each of which is compact by assumption. As the union of two compact spaces is again compact, we conclude that $L \cup K$ is indeed a subset of Y with compact closure. Hence, $U \cap V \in \mathcal{U}_p$. $\square$

Claim. *If $U \in \mathcal{U}_p$, then there is a set $V \in \mathcal{U}_p$ such that $U \in \mathcal{U}_y$ for each $y \in V$.*

Proof. Let $U \in \mathcal{U}_p$ and choose any open $V \in \mathcal{U}_p$ with $V \subset U$. Since V is open, there exists, for all $y \in V$, an open subset G of V with $y \in G$ and $p \notin G$. Hence, $G \in \mathcal{U}_y$. Since $y \in Y$, $\mathcal{U}_y$ satisfies the property that any superset of G is also contained in $\mathcal{U}_y$. In particular, $U \in \mathcal{U}_y$, as desired. $\square$

Claim. *If $U \in \mathcal{U}_p$ and $U \subset V$, then $V \in \mathcal{U}_p$.*

Proof. Let $U = \{p\} \cup (Y - L)$ for some subset L of Y with compact closure. It follows that $V = \{p\} \cup (Y - K)$ with $K \subset L$, which in turn gives that $\mathrm{Cl}_Y(K) \subset \mathrm{Cl}_Y(L)$. As a closed subset of compact space is compact, we have that K has compact closure. Hence, $V \in \mathcal{U}_p$. $\square$

Therefore, the neighborhood systems as defined indeed give a topology on Y^*. We conclude by showing that its relative topology on Y is the original topology. To that end, let $\sigma = \{G \mid G \text{ open in } Y\}$ and $\tau = \{G \cap Y \mid G \text{ open in } Y^*\}$. Evidently, $\sigma \subset \tau$. We also have that, for any open G in Y^*,

$$\begin{aligned} G \cap Y &= [\{p\} \cup (Y - L)] \cap Y \qquad \text{(some } L \text{ with compact closure in } Y) \\ &= Y - L. \end{aligned}$$

(Do we here require the stronger notion that L be compact rather than merely having compact closure? For example, $(0,1)$ has compact closure in $\mathbb{R}$, yet $\mathbb{R} - (0,1)$ is not open.)

Pending the resolution of the above parenthetical remark, we will have shown that $\sigma = \tau$, and so the relative topology on Y is precisely the original topology. $\square$

3 Problem 19A4

Proposition 3.1. *Y^* is Hausdorff if and only if Y is locally compact and Hausdorff.*

Proof. ($\Rightarrow$) Let $x, y \in Y \subset Y^*$. Since Y^* is Hausdorff, we can find disjoint open neighborhoods U and V in Y^* with $x \in U$ and $y \in V$. If neither U nor V contains p, then U and V suffice to show that Y is Hausdorff. Otherwise, let it be that $U = \{p\} \cup (Y - L)$ for some L with compact closure in Y (it cannot be that both U and V are of this form, as this would violate disjointness). Define the set $U' = Y - Cl_Y(L)$. We have that $x \in U'$, $y \in V$ with U' and V disjoint and open.

To see that Y is locally compact, let $x \in Y$ be given. Since Y^* is Hausdorff, there is an open set U of Y^* such that $x \notin U$. Now, $U = \{p\} \cup (Y - L)$ for some subset L of Y with compact closure. Hence, $x \in L \subset \mathrm{Cl}_Y(L)$, which is compact. Since Y is Hausdorff and every point of Y has a compact neighborhood, it follows that Y is locally compact.

($\Leftarrow$) Let $x, y \in Y^*$. If neither x nor y is p, then the fact that Y is Hausdorff implies that x and y can be put into disjoint open neighborhoods. Suppose now that $y = p$. Choose, by the local compactness of Y, some compact neighborhood $V \subset Y$ of x. It follows that $\mathrm{Int}_Y(V)$ is an open set containing x. We also have that $\mathrm{Int}_Y(Y^* - V) = \mathrm{Int}_Y(\{p\} \cup (Y - V))$, which is an open neighborhood (as it is a member of the base) containing p. Thus, for any $x, y \in Y^*$, we have produced two disjoint open sets containing them. Hence, Y^* is Hausdorff. $\square$