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Math 740
Homework 0
(Galois Connections)

In the problems below, let A and B be two classes and let R be a binary relation with $R \subseteq A \times B$. For $X \subseteq A$ and $Y \subseteq B$, put

$$X^{\rightarrow} = \{b \mid xRb \text{ for all } x \in X\}$$
$$Y^{\leftarrow} = \{a \mid aRy \text{ for all } y \in Y\}$$

Problem 0

Prove that if $W \subseteq X \subseteq A$, then $X^{\rightarrow} \subseteq W^{\rightarrow}$. (Likewise, if $V \subseteq Y \subseteq B$, then $Y^{\leftarrow} \subseteq V^{\leftarrow}$.)

Proof.

$$\begin{aligned} y \in X^{\rightarrow} &\Rightarrow aRy \text{ for all } a \in X \\ &\Rightarrow aRy \text{ for all } a \in W \text{ (as } W \subseteq X) \\ &\Rightarrow y \in W^{\rightarrow} \end{aligned}$$

Therefore, $X^{\rightarrow} \subseteq W^{\rightarrow}$.

Similarly,

$$\begin{aligned} x \in Y^{\leftarrow} &\Rightarrow xRb \text{ for all } b \in Y \\ &\Rightarrow xRb \text{ for all } b \in V \text{ (as } V \subseteq Y) \\ &\Rightarrow x \in V^{\leftarrow} \end{aligned}$$

Therefore, $Y^{\leftarrow} \subseteq V^{\leftarrow}$. □

Problem 1

Prove that if $X \subseteq A$, then $X \subseteq X^{\rightarrow\leftarrow}$. (Likewise, if $Y \subseteq B$, then $Y \subseteq Y^{\leftarrow\rightarrow}$.)

Proof. Let $x \in X$. By definition,

$$X^{\rightarrow} = \{b \mid aRb \text{ for all } a \in X\}$$

Hence, xRb for all $b \in X^{\rightarrow}$. Also by definition, we have that

$$X^{\rightarrow\leftarrow} = \{a \mid aRb \text{ for all } b \in X^{\rightarrow}\},$$

and so $x \in X^{\rightarrow\leftarrow}$. Therefore, $X \subseteq X^{\rightarrow\leftarrow}$.

Similarly, let $y \in Y$. By definition,

$$Y^{\leftarrow} = \{a \mid aRb \text{ for all } b \in Y\}$$

Hence, aRy for all $a \in Y^{\leftarrow}$. Also by definition, we have that

$$Y^{\leftarrow\rightarrow} = \{b \mid aRb \text{ for all } a \in Y^{\leftarrow}\},$$

and so $y \in Y^{\leftarrow\rightarrow}$. Therefore, $Y \subseteq Y^{\leftarrow\rightarrow}$. □

Problem 2

Prove that $X^{\rightarrow\leftarrow\rightarrow} = X^{\rightarrow}$ for all $X \subseteq A$. (Likewise, $Y^{\leftarrow\rightarrow\leftarrow} = Y^{\leftarrow}$ for all $Y \subseteq B$.)

Proof. As $X \subseteq A$, problem 1 gives that

$$X \subseteq X^{\rightarrow\leftarrow}.$$

Combining this with the result of problem 0, we see that

$$X^{\rightarrow\leftarrow\rightarrow} \subseteq X^{\rightarrow}.$$

Now, as $X^{\rightarrow} \subseteq B$, problem 1 gives that

$$X^{\rightarrow} \subseteq X^{\rightarrow\leftarrow\rightarrow}.$$

Therefore, $X^{\rightarrow\leftarrow\rightarrow} = X^{\rightarrow}$.

Similarly, as $Y \subseteq B$, problem 1 gives that

$$Y \subseteq Y^{\leftarrow\rightarrow}.$$

Combining this with the result of problem 0, we see that

$$Y^{\leftarrow\rightarrow\leftarrow} \subseteq Y^{\leftarrow}.$$

Now, as $Y^{\leftarrow} \subseteq A$, problem 1 gives that

$$Y^{\leftarrow} \subseteq Y^{\leftarrow\rightarrow\leftarrow}.$$

Therefore, $Y^{\leftarrow\rightarrow\leftarrow} = Y^{\leftarrow}$. □

Problem 3

Prove that the collection of subclasses of A of the form $Y^{\leftarrow}$ is closed under the formation of arbitrary intersections (as is the collection of subclasses of B of the form $X^{\rightarrow}$). We call classes of the form $Y^{\leftarrow}$ and of the form $X^{\rightarrow}$ closed.

Proof. Let $\{Y_\alpha \mid \alpha \in I\}$ be a family of subsets of B for some indexing set I. If $\bigcap_{\alpha\in I} Y_\alpha^{\leftarrow}$ is empty, then the claim holds trivially, as

$$\bigcap_{\alpha\in I} Y_\alpha^{\leftarrow} = \emptyset^{\leftarrow}.$$

Otherwise,

$$\begin{aligned} x \in \bigcap_{\alpha\in I} Y_\alpha^{\leftarrow} \quad &\Leftrightarrow \quad x \in Y_\alpha^{\leftarrow} \text{ for all } \alpha \in I \\ &\Leftrightarrow \quad xRb \text{ for all } b \in Y_\alpha \text{ for all } \alpha \in I \\ &\Leftrightarrow \quad xRb \text{ for all } b \in \bigcup_{\alpha\in I} Y_\alpha \\ &\Leftrightarrow \quad x \in \left(\bigcup_{\alpha\in I} Y_\alpha\right)^{\leftarrow} \end{aligned}$$

Therefore, $\bigcap_{\alpha\in I} Y_\alpha^{\leftarrow} = \left(\bigcup_{\alpha\in I} Y_\alpha\right)^{\leftarrow}$, and so is closed.

Similarly, let $\{X_\alpha \mid \alpha \in I\}$ be a family of subsets of A for some indexing set I. If $\bigcap_{\alpha\in I} X_\alpha^{\rightarrow}$ is empty, then the claim holds trivially, as

$$\bigcap_{\alpha\in I} X_\alpha^{\rightarrow} = \emptyset^{\rightarrow}.$$

Otherwise,

$$\begin{aligned} y \in \bigcap_{\alpha \in I} X_\alpha^{\rightarrow} &\Leftrightarrow y \in X_\alpha^{\rightarrow} \text{ for all } \alpha \in I \\ &\Leftrightarrow aRy \text{ for all } a \in X_\alpha \text{ for all } \alpha \in I \\ &\Leftrightarrow aRy \text{ for all } a \in \bigcup_{\alpha \in I} X_\alpha \\ &\Leftrightarrow y \in \left(\bigcup_{\alpha \in I} X_\alpha \right)^{\rightarrow} \end{aligned}$$

Therefore, $\bigcap_{\alpha \in I} X_\alpha^{\rightarrow} = \left(\bigcup_{\alpha \in I} X_\alpha \right)^{\rightarrow}$, and so is closed. □

Problem 4

Let $A = B = \{q \mid 0 < q < 1 \text{ and } q \text{ is rational}\}$. Let R be the binary operation $\leq$. Identify the system of closed sets. How are they ordered with respect to inclusion?

Proof. Let $X \subseteq A$. Since R is $\leq$,

$$X^{\rightarrow} = \{b \mid x \leq b \text{ for all } x \in X\}.$$

Hence, $X^{\rightarrow}$ is the set of all upper bounds of X. By definition, $\sup X$ is the least upper bound of X, so we can describe the set of all upper bounds of X by

$$X^{\rightarrow} = \{b \in B \mid \sup X \leq b\}.$$

Observe further that, for any $a \in A$, the set $X = \{a\}$ has the property that $\sup X = a$. Additionally, the set $X = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$ has $\sup X = 1$, while no subset X of A has $\sup X = 0$. Therefore, the class of closed sets of the form $X^{\rightarrow}$ is precisely

$$\{(a, 1) \cap \mathbb{Q} \mid a \in (0, 1)\} \cup \emptyset$$

Similarly, let $Y \subseteq B$. Since R is $\leq$,

$$Y^{\leftarrow} = \{a \mid a \leq y \text{ for all } y \in Y\}.$$

Hence, $Y^{\leftarrow}$ is the set of all lower bounds of Y. By definition, $\inf Y$ is the greatest lower bound of Y, so we can describe the set of all lower bounds of Y by

$$Y^{\leftarrow} = \{a \in A \mid a \leq \inf Y\}.$$

Observe further that, for any $b \in B$, the set $Y = \{b\}$ has the property that $\inf Y = b$. Additionally, the set $Y = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ has $\inf Y = 0$, while no subset Y of B has $\inf Y = 1$. Therefore, the class of closed sets of the form $Y^{\leftarrow}$ is precisely

$$\{(0, b) \cap \mathbb{Q} \mid b \in (0, 1)\} \cup \emptyset$$

□