# Math 731 Homework 11

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Define $\sim$ in any space $X$ by $x \sim y$ if and only if $x$ and $y$ lie together in some connected subset of $X$. Define $\approx$ in $X$ by $x \approx y$ if and only if there is no decomposition $X = U \cup V$ into disjoint open sets with one containing $x$ and the other containing $y$.

## 1 Problem 26B1

**Proposition 1.1.** *The relation $\sim$ is an equivalence on $X$. The equivalence class $[x]$ of $x$ is just the component $C_x$ of $x$ in $X$.*

*Proof.* The relation $\sim$ is reflexive, as $x$ lies within the connected subset of $X$ containing it (that is, $x \sim x$).

The relation $\sim$ is symmetric, as "$x$ and $y$ lie together in some connected subset" has the same meaning as "$y$ and $x$ lie together in some connected subset".

To see that $\sim$ is transitive, suppose that $x \sim y$ and $y \sim z$ for some $x, y, z \in X$. Let $U$ denote the connected subset of $X$ containing $x$ and $y$, and let $V$ denote the connected subset of $X$ containing $y$ and $z$. We see that $U \cap V \neq \emptyset$ (as $y \in U \cap V$), and so $U \cup V$ is connected. Hence, $x \sim z$, as they lie together in the connected subset $U \cup V$ of $X$.

Taken together, we conclude that $\sim$ is an equivalence relation on $X$.

Now, let $y \in [x]$. This means that $x$ and $y$ lie in some connected subset of $X$. Since $C_x$ is the union of *all* connected subsets of $X$ containing $x$, we see that $y \in C_x$. Next, let $y \in C_x$. Since $x$ and $y$ both lie in the connected component $C_x$, we see that $y \in [x]$. Hence, $[x] = C_x$, as desired. □

## 2 Problem 26B2

**Proposition 2.1.** *The relation $\approx$ is an equivalence on $X$. We call the equivalence class of $x$ the* quasicomponent *of $x$ in $X$. The quasicomponent of $x$ in $X$ is the intersection of all clopen subsets of $X$ that contain $x$.*

*Proof.* The relation $\approx$ is reflexive, since there can be no decomposition separating $x$ from itself (that is, $x \sim x$).

The relation $\approx$ is symmetric, as $U$ and $V$ are indistinguishable in the definition. That is, the phrase "one containing $x$ and the other containing $y$" has the same meaning as "one containing $y$ and the other containing $x$".

To see that $\approx$ is transitive, we establish the contrapositive. To that end, suppose that $x \not\approx z$. This means that there are disjoint open sets $U$ and $V$ such that (without loss of generality) $x \in U$, $z \in V$, and $X = U \cup V$. Now, it must be that $y$ belongs to one of $U$ and $V$. If $y \in U$, then $U$ and $V$ represent a decomposition of $X$ of the appropriate type separating $y$ from $z$, and so $y \not\approx z$. Similarly, if $y \in V$, then $U$ and $V$ separate $x$ from $y$, and so $x \not\approx y$. Hence, we have that $x \not\approx z \Rightarrow (x \not\approx y) \vee (y \not\approx z)$, thus establishing the contrapositive.

Taken together, we conclude that $\approx$ is an equivalence relation on $X$.

Let now $F$ denote the intersection of all clopen subsets of $X$ containing $x$. Let $y \in F$. This means that $y$ belongs to *every* clopen subset of $X$ containing $x$. In particular, $y \in C_x$. Since $C_x$ is connected, we see that $x$ cannot be separated from $y$ by a pair of disjoint open sets whose union is $X$ (otherwise, these sets would disconnect $C_x$). Hence, $y \in [x]$. Next, we show that $[x] \subset F$ by establishing the contrapositive. To that end, suppose that $y \notin F$. This means there is some clopen subset $F_y$ of $X$ such that $x \in F_y$ but $y \notin F_y$. It follows that $F_y$ and $F_y^c$ are disjoint open sets with $x \in F_y$, $y \in F_y^c$, and $X = F_y \cup F_y^c$. Hence, $y \notin [x]$, thus establishing the contrapositive. Therefore, $[x] = F$, as desired. □

(I could make no sense of Willard's picture, so I provide a different space where the components and quasicomponents may disagree.)

# 3 Extra Problem

**Proposition 3.1.** *Let $A$ denote the set $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ and let $X$ be the space $(A \times [0,1]) \cup \{(0,0)\} \cup \{(0,1)\}$ with the relative Euclidean topology. The points $\{(0,0)\}$ and $\{(0,1)\}$ belong to separate components, but belong to the same quasicomponent.*

*Proof.* Evidently, $C_{(0,0)} = \{(0,0)\}$, since $\{(0,0)\}$ is the *only* connected subset of $X$ containing $(0,0)$. We claim next that $\{(0,0),(0,1)\} \subset [(0,0)]$. Suppose, for the purpose of contradiction, that there are disjoint open sets $U$ and $V$ such that (without loss of generality) $(0,0) \in U$, $(0,1) \in V$, and $U \cup V = X$. Since $U$ is open, $(0,0) \subsetneq U$. Similarly, $(0,1) \subsetneq V$. Let $k$ be a natural number such that both $U \cap (\frac{1}{k} \times [0,1]) \neq \emptyset$ and $V \cap (\frac{1}{k} \times [0,1]) \neq \emptyset$. Now, as $\frac{1}{k} \times [0,1]$ is a connected subset of $X$, it must belong entirely to $U$ or $V$, contradicting our previous assertion that *both* $U$ and $V$ intersect it nontrivially. Therefore, $(0,0)$ and $(0,1)$ belong to separate components, yet they belong to the same quasicomponent. $\square$