Math 731 Homework 10 (Correction 1)

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1 Problem 17F1

Proposition 1.1. *A space is countably compact if and only if each sequence has a cluster point.*

Proof. ($\Rightarrow$) We proceed by establishing the contrapositive. To that end, suppose there is a sequence $\langle x_n \rangle$ having no cluster point in X. That is, for every $x \in X$, there exists a neighborhood (without loss of generality, an *open* neighborhood) N_x of x such that $N_x \setminus \{x\}$ contains no point of $\langle x_n \rangle$. Denote by N the open set $\bigcup \{N_x \mid x \text{ is not an element } \langle x_n \rangle\}$. By construction, N contains no element of the sequence $\langle x_n \rangle$. Finally, let $\mathcal{C}$ denote the collection of open sets $\{N_x \mid x \text{ is an element of } \langle x_n \rangle\} \cup \{N\}$. We see that $\mathcal{C}$ is a countable cover of X that admits no finite subcover. Indeed, omitting *any* N_x results in a subcollection missing x. Therefore, X is not countably compact, thus establishing the contrapositive.

($\Leftarrow$) Let X be a space with the property that every sequence has a cluster point. Take any family of closed sets C_n having the finite intersection property and let x_n be any point belonging to $\bigcap_{i=1}^{n} C_i$. Let x be a cluster point of the sequence $\langle x_n \rangle$ and let O be any open set containing x. Since x is a cluster point, O contains infinitely-many x_n, and hence O intersects every C_n. As O was arbitrary, we conclude that x belongs to every C_n. Hence, $x \in \bigcap_{n=1}^{\infty} C_n$, and so X is countably compact. ☐

2 Problem 20B

Lemma 2.1. *Let $\mathcal{U}$ be a collection of open sets. For all $U \in \mathcal{U}$,*

$$\mathrm{St}(U,\mathcal{U}) = \bigcup_{x \in U} \mathrm{St}(x,\mathcal{U}).$$

Proof. (To be included if the lemma turns out to be useful) ☐

Proposition 2.2. *A barycentric refinement of a barycentric refinement of a cover $\mathcal{W}$ is a star-refinement of $\mathcal{W}$.*

Proof. Let $\mathcal{U}\Delta\mathcal{V}\Delta\mathcal{W}$. Our goal is to show that $\mathcal{U} \;^{*}\!< \mathcal{W}$. To that end, let $U \in \mathcal{U}$. Now,

$$\begin{aligned}
\mathrm{St}(U,\mathcal{U}) &= \bigcup_{x \in U} \mathrm{St}(x,\mathcal{U}) && \text{(by lemma)}\\
&\subset \bigcup_{x \in U} V_x, \text{ for some } V_x \in \mathcal{V} && \text{(since } \mathcal{U}\Delta\mathcal{V}\text{)}\\
&\subset \bigcup_{x \in U} \mathrm{St}(x,\mathcal{V})\\
&\subset \bigcup_{x \in U} W_x, \text{ for some } W_x \in \mathcal{W} && \text{(since } \mathcal{V}\Delta\mathcal{W}\text{)}.
\end{aligned}$$

(My problem now is that there is no guarantee that $\bigcup_{x \in U} W_x \in \mathcal{W}$, nor can I force $\bigcup_{x \in U} \mathrm{St}(x,\mathcal{V}) \subset \mathrm{St}(x_0,\mathcal{V})$ for any single x_0.) ☐

Proposition 2.3. *If $\mathcal{U}_n$ is the cover of a metric space (X,d) by $\left(\frac{1}{3^n}\right)$-balls about each of its points, then $\mathcal{U}_{n+1} \;^{*}\!< \mathcal{U}_n$.*

Proof. Let $U \in \mathcal{U}_{n+1}$. By definition, $U = B(x, \frac{1}{3^{n+1}})$ for some $x \in X$. Consider now any point $y \in \mathrm{St}(U,\mathcal{U}_{n+1})$. Now, $y \in V$ for some $V \in \mathcal{U}_{n+1}$ with $U \cap V \neq \emptyset$. It follows that $d(x,y) < \frac{3}{3^{n+1}} = \frac{1}{3^n}$, as x is the center of a $\left(\frac{1}{3^{n+1}}\right)$-ball and V has diameter $\frac{2}{3^{n+1}}$. Hence, $\mathrm{St}(U,\mathcal{U}_{n+1}) \subset B(x, \frac{1}{3^n}) \in \mathcal{U}_n$, as desired. ☐

Proposition 2.4. *If $\mathcal{U}$ is an open cover of X, $\mathcal{V}$ is an open barycentric refinement of $\mathcal{U}$, and for each $U \in \mathcal{U}$ we define $F_U = X - St(X - U, \mathcal{V})$, then $\{F_U \mid U \in \mathcal{U}\}$ is a closed cover of X.*

Proof. The fact that each F_U is closed is immediate, since it is the complement of a union of open sets. Let now $x \in X$. Since $\mathcal{V}\Delta\mathcal{U}$, $\mathrm{St}(x, \mathcal{V}) \subset U$, for some $U \in \mathcal{U}$. We claim that $x \in F_U$ for this U. Suppose, for the purpose of contradiction, that this is not the case. This means that $x \in \mathrm{St}(X - U, \mathcal{V})$. In particular, there exists some $V \in \mathcal{V}$ with $x \in V$ and $V \cap (X - U) \neq \emptyset$. On the other hand, V is an open set containing x, and so $V \subset \mathrm{St}(x, \mathcal{V}) \subset U$. Having arrived at a contradiction (V cannot be both a subset of U and also intersect $X - U$ nontrivially), we conclude that $x \in F_U$. As x was chosen arbitrarily, we see that $\{F_U \mid U \in \mathcal{U}\}$ is indeed a closed cover of X, as desired. $\square$