Austin Mohr
Math 701
Homework 1

For the following problems, let

$$\mathbf{A} = \langle A, *_i \rangle$$
$$\mathbf{B} = \langle B, \Box_i \rangle$$

be algebras with r-ary operations for each i in some indexed set I. Then, we have the algebra

$$\mathbf{A} \times \mathbf{B} = \langle A \times B, \star_i \rangle$$

where

$$\star_i((a_0, b_0), \ldots, (a_{r-1}, b_{r-1})) = (*_i(a_0, \ldots, a_{r-1}), \Box_i(b_0, \ldots, b_{r-1}))$$

with $a_j \in A$ and $b_j \in B$ for $0 \leq j \leq r-1$.

Problem 3

Prove that the congruence relations of $\mathbf{A}$ are exactly those subuniverses of $\mathbf{A} \times \mathbf{A}$ which happen to be equivalence relations on A.

Proof. ($\Rightarrow$) Let Θ be an equivalence relation on A. Define the algebra $\mathbf{T} = \langle T, \star_i \rangle$ for all $i \in I$ where

$$T = \{(a, a') \mid a, a' \in A, a\Theta a'\}$$

It is clear that $T \subseteq A \times A$ and defines an equivalence relation (since Θ is an equivalence relation). To see that it preserves $\star_i$, let $(a_0, a'_0), \ldots, (a_{r-1}, a'_{r-1}) \in T$. Then, for each $i \in I$

$$\star_i((a_0, a'_0), \ldots, (a_{r-1}, a'_{r-1})) = (*_i(a_0, \ldots, a_{r-1}), *_i(a'_0, \ldots, a'_{r-1}))$$

Now, since Θ is a congruence relation, we have that

$$*_i(a_0, \ldots, a_{r-1})\Theta *_i (a'_0, \ldots, a'_{r-1})$$

and so

$$(*_i(a_0, \ldots, a_{r-1}), *_i(a'_0, \ldots, a'_{r-1})) \in T$$

($\Leftarrow$) Let $\mathbf{E} = \langle E, \star_i \rangle$ for all $i \in I$ be an algebra where

$$E = \{(a, a') \mid a, a' \in A, a \text{ is equivalent to } a'\}$$

and so $\mathbf{E}$ is a subuniverse of $\mathbf{A} \times \mathbf{A}$. Define the relation Θ

$$a\Theta a' \text{ if and only if } (a, a') \in E.$$

It is clear that Θ is an equivalence relation (since E is an equivalence relation). To see that Θ preserves $\star_i$, suppose $a_0\Theta a'_0, \ldots, a_{r-1}\Theta a'_{r-1}$. Then, $(a_0, a'_0), \ldots, (a_{r-1}, a'_{r-1}) \in E$. Hence, for all $i \in I$

$$\star_i((a_0, a'_0), \ldots, (a_{r-1}, a'_{r-1})) = (*_i(a_0, \ldots, a_{r-1}), *_i(a'_0, \ldots, a'_{r-1}))$$

Since E is closed under $\star_i$, we have that

$$(*_i(a_0, \ldots, a_{r-1}), *_i(a'_0, \ldots, a'_{r-1})) \in E$$

and so

$$*_i(a_0, \ldots, a_{r-1})\Theta *_i (a'_0, \ldots, a'_{r-1})$$

□

Problem 4

Prove that the homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ are exactly those subuniverses of $\mathbf{A} \times \mathbf{B}$ which are functions from A to B.

Proof. (⇒) Let $h : \mathbf{A} \to \mathbf{B}$ be a homomorphism. Define the algebra $\mathbf{H} = \langle H, \star_i \rangle$ for all $i \in I$ where

$$H = \{(a, h(a)) \mid a \in A\}$$

It is clear that $H \subseteq A \times B$ and defines a function from A to B (namely h). To see that it preserves $\star_i$, let $a_0, \ldots, a_{r-1} \in A$. Then, for each $i \in I$

$$\begin{aligned}
\star_i((a_0, h(a_0)), \ldots, (a_{r-1}, h(a_{r-1}))) &= (*_i(a_0, \ldots, a_{r-1}), \Box_i(h(a_0), \ldots, h(a_{r-1}))) \\
&= (*_i(a_0, \ldots, a_{r-1}), h(\Box_i(a_0, \ldots, a_{r-1}))) \\
&\in H
\end{aligned}$$

(⇐) Let $\mathbf{F} = \langle F, \star_i \rangle$ for all $i \in I$ be an algebra where

$$F = \{(a, f(a)) \mid a \in A\}$$

for some function $f : A \to B$ (and so $\mathbf{F}$ is a subuniverse of $\mathbf{A} \times \mathbf{B}$). Since $\mathbf{F}$ is an algebra, we have for $a_0, \ldots, a_{r-1} \in A$ and for each $i \in I$

$$\star_i((a_0, f(a_0)), \ldots, (a_{r-1}, f(a_{r-1}))) = (*_i(a_0, \ldots, a_{r-1}), \Box_i(f(a_0), \ldots, f(a_{r-1}))) \in F$$

and so

$$f(*_i(a_0, \ldots, a_{r-1})) = \Box_i(f(a_0), \ldots, f(a_{r-1}))$$

In other words, f is a homomorphism. □

<u>Problem 5</u>

Prove that the projection functions associated with $\mathbf{A} \times \mathbf{B}$ are homomorphisms.

Proof. Define

$$\pi_0 : A \times B \to A$$
$$\pi_0(a, b) = a \text{ for all } a \in A \text{ and } b \in B$$

Now, let $a_0, \ldots, a_{r-1} \in A$ and $b_0, \ldots, b_{r-1} \in B$. Then, for each $i \in I$,

$$\begin{aligned}
\pi_0(\star_i((a_0, b_0), \ldots, (a_{r-1}, b_{r-1}))) &= \pi_0((*_i(a_0, \ldots, a_{r-1}), \Box_i(b_0, \ldots, b_{r-1}))) \\
&= *_i(a_0, \ldots, a_{r-1}) \\
&= *_i(\pi_0((a_0, b_0)), \ldots, \pi_0((a_{r-1}, b_{r-1})))
\end{aligned}$$

Similarly, define

$$\pi_1 : A \times B \to B$$
$$\pi_1(a, b) = b \text{ for all } a \in A \text{ and } b \in B$$

Then, for all $i \in I$

$$\begin{aligned}
\pi_1(\star_i((a_0, b_0), \ldots, (a_{r-1}, b_{r-1}))) &= \pi_1((*_i(a_0, \ldots, a_{r-1}), \Box_i(b_0, \ldots, b_{r-1}))) \\
&= \Box_i(b_0, \ldots, b_{r-1}) \\
&= \Box_i(\pi_1((a_0, b_0)), \ldots, \pi_1((a_{r-1}, b_{r-1})))
\end{aligned}$$

□