Austin Mohr
Math 701
Homework 0

Problem 0

Classify up to similarity all the square matrices over the complex numbers with minimal polynomial $m(x) = (x-1)^2(x-2)^2$ and characteristic polynomial $c(x) = (x-1)^6(x-2)^5$.

Let A be a matrix of the desired form. Observe first that, since we are working over $\mathbb{C}$, A can be expressed in Jordan form. Now, the roots of $c(x)$ are precisely the eigenvalues of A. Furthermore, a root's multiplicity in $m(x)$ gives the size of the largest elementary Jordan block in A. Since, up to similarity, permutations of the blocks are irrelevant, we have the following possibilities for A:

$$\begin{pmatrix}
1 & 1 \\
0 & 1 \\
& & 1 & 1 \\
& & 0 & 1 \\
& & & & 1 & 1 \\
& & & & 0 & 1 \\
& & & & & & 2 & 1 \\
& & & & & & 0 & 2 \\
& & & & & & & & 2 & 1 \\
& & & & & & & & 0 & 2 \\
& & & & & & & & & & 2
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 1 \\
0 & 1 \\
& & 1 & 1 \\
& & 0 & 1 \\
& & & & 1 & 1 \\
& & & & 0 & 1 \\
& & & & & & 2 & 1 \\
& & & & & & 0 & 2 \\
& & & & & & & & 2 \\
& & & & & & & & & 2 \\
& & & & & & & & & & 2
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 1 \\
0 & 1 \\
& & 1 & 1 \\
& & 0 & 1 \\
& & & & 1 \\
& & & & & 1 \\
& & & & & & 2 & 1 \\
& & & & & & 0 & 2 \\
& & & & & & & & 2 & 1 \\
& & & & & & & & 0 & 2 \\
& & & & & & & & & & 2
\end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & & \\ & & 1 & 1 & & & & & & & \\ & & 0 & 1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & 1 & & & & & \\ & & & & & & 2 & 1 & & & \\ & & & & & & 0 & 2 & & & \\ & & & & & & & & 2 & & \\ & & & & & & & & & 2 & \\ & & & & & & & & & & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & 1 & & & & & \\ & & & & & & 2 & 1 & & & \\ & & & & & & 0 & 2 & & & \\ & & & & & & & & 2 & 1 & \\ & & & & & & & & 0 & 2 & \\ & & & & & & & & & & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & 1 & & & & & \\ & & & & & & 2 & 1 & & & \\ & & & & & & 0 & 2 & & & \\ & & & & & & & & 2 & & \\ & & & & & & & & & 2 & \\ & & & & & & & & & & 2 \end{pmatrix}$$

Problem 1

Let $T : V \to V$ be a linear transformation of rank 1 on a finite dimensional vector space V over any field. Prove that either T is nilpotent or V has a basis of eigenvectors of T.

Proof. Let $dim(V) = n$ and let $\{\vec{v_0}\}$ be a basis for $R(T)$ (since $rank(T) = 1$). Observe that $\vec{v_0} = T(\vec{v_0'})$ for some $\vec{v_0'} \in V$. By the dimension theorem, the nullity of T is $n-1$, and so let $\{\vec{w_0}, \cdots, \vec{w_{n-2}}\}$ be a basis for $N(T)$.

Observe that, since $T \neq 0$, it cannot be that T is nilpotent *and* V has a basis of eigenvectors of T.

Suppose T is not nilpotent. We claim that $\{\vec{v_0}, \vec{w_0}, \cdots, \vec{w_1}\}$ is a basis of eigenvectors for V. Since T is not nilpotent, we have that $T(\vec{v_0}) = T^2(\vec{v_0'}) \neq 0$. Since T has rank 1, it must be that $T(\vec{v_0}) = \lambda \vec{v_0}$ for some $\lambda \neq 0$. In other words, $\vec{v_0}$ is an eigenvector for T. Now, $T(\vec{w_i}) = 0\vec{w_i}$ for all i, and so each $\vec{w_i}$ is an eigenvector for T with eigenvalue 0.

Now, the set $\{\vec{w_0}, \cdots, \vec{w_{n-2}}\}$ is independent, since it is a basis for $N(T)$. Suppose $\vec{v_0}$ can be expressed as a linear combination of the $\vec{w_i}$. Then

$$\begin{aligned}
\vec{v_i} &= \sum_{i=0}^{n-2} a_i \vec{w_i} \\
T(\vec{v_i}) &= T(\sum_{i=0}^{n-2} a_i \vec{w_i}) \\
T(\vec{v_i}) &= \sum_{i=0}^{n-2} a_i T(\vec{w_i}) \\
T(\vec{v_i}) &= 0
\end{aligned}$$

which is a contradiction. Hence, $\{\vec{v_0}, \vec{w_0}, \cdots, \vec{w_{n-2}}\}$ is an independent set of n eigenvectors of T, and so a basis for V. $\square$

Problem 2

Let V be a vector space over a field K.

a. Prove that if U_0 and U_1 are subspaces of V such that $U_0 \not\subseteq U_1$ and $U_1 \not\subseteq U_0$, then $V \neq U_0 \cup U_1$.

Proof. Since $U_0 \not\subseteq U_1$, there exists a basis vector $U_0 \not\ni \vec{u_0} \in U_1$. Similarly, since $U_1 \not\subseteq U_0$, there exists a basis vector $U_0 \ni \vec{u_1} \notin U_1$. Now, consider the linear combination $\vec{u_0} + \vec{u_1} \in V$.

$\vec{u_0} + \vec{u_1} \notin U_0$ (since $\vec{u_0} \notin U_0$)

$\vec{u_0} + \vec{u_1} \notin U_1$ (since $\vec{u_1} \notin U_1$)

Hence, $\vec{u_0} + \vec{u_1} \notin U_0 \cup U_1$, and so $U_0 \cup U_1 \neq V$. □

b. Prove that if U_0, U_1, and U_2 are subspaces of V such that $U_i \not\subseteq U_j$ when $i \neq j$ and K has at least 3 elements, then $V \neq U_0 \cup U_1 \cup U_2$.

Proof. Since no subspace is contained in any other, we can find basis vectors $\vec{u_0} \notin U_0$, $\vec{u_1} \notin U_1$, and $\vec{u_2} \notin U_2$, $\vec{u_i} \in V$. Observe that if all the $\vec{u_i}$ are identical, then we have found a vector in V which is in none of the U_i, and so $U_0 \cup U_1 \cup U_2 \neq V$. Let it be, instead, that at least one of the $\vec{u_i}$ differs from the rest. Without loss of generality, let $\vec{u_0}$ differ from the rest of the $\vec{u_i}$. Now, consider the linear combination $\vec{u_0} + \vec{u_1} + \vec{u_2} \in V$. It is possible that $\vec{u_1} = \vec{u_2}$, but since K is of characteristic 3, we are assured that $\vec{u_1} + \vec{u_2} \neq 0$. It follows that, for all i $\quad \vec{u_0} + \vec{u_1} + \vec{u_2} \notin U_i$ (since $\vec{u_i} \notin U_i$)

Hence, $\vec{u_0} + \vec{u_1} + \vec{u_2} \notin U_0 \cup U_1 \cup U_2$, and so $U_0 \cup U_1 \cup U_2 \neq V$. □

c. State a prove a generalization of (b) for n subspaces.

Proposition 0.1. *Let V be a finite-dimensional vector space over a field K of characteristic at least n. Let $\{U_i \mid 0 \leq i \leq n-1\}$ be a set of subspaces such that $U_i \not\subseteq U_j$ for $i \neq j$. Then,*

$$\bigcup_{i=0}^{n-1} U_i \neq V$$

Proof. Since no subspace is contained in any other, we can find basis vectors $V \ni \vec{u_i} \notin U_i$ for each $0 \leq i \leq n-1$. Observe that if all the $\vec{u_i}$ are identical, then we have found a vector in V which is in none of the U_i, and so $\bigcup_{i=0}^{n-1} U_i \neq V$. Let it be, instead, that at least one of the $\vec{u_i}$ differs from the rest. Without loss of generality, let $\vec{u_0}$ differ from the rest of the $\vec{u_i}$. Now, consider the linear combination $\sum_{i=0}^{n-1} \vec{u_i} \in V$. It is possible that $\vec{u_1} = \vec{u_2} = \cdots = \vec{u_{n-1}}$, but since K is of characteristic n, we are assured that $\sum_{i=1}^{n-1} \vec{u_i} \neq 0$. It follows that, for all $0 \leq j \leq n-1$

$$\sum_{i=0}^{n-1} \vec{u_i} \notin U_j \text{ (since } \vec{u_j} \notin U_j)$$

Hence,

$$\sum_{i=0}^{n-1} \vec{u_i} \notin \bigcup_{i=0}^{n-1} U_i$$

and so $\bigcup_{i=0}^{n-1} U_i \neq V$. □