Math 778G Homework 3

Austin Mohr

May 4, 2011

Problem 1

Proposition 1. *For $n \geq d$ and $d \geq 4$, the cyclic polytope with n vertices in $\mathbb{R}^d$ has the property that every two vertices are on an edge together.*

Proof. Let v_1 and v_2 be any two vertices of the d-dimensional cyclic polytope and let t_1 and t_2 be the corresponding moment curve parameters. Consider the hyperplane H defined by

$$a_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

where each a_i is the coefficient of t^i in the polynomial $p(t) = (t - t_1)^2(t - t_2)^2$. As t_1 is a root of p, we have

$$a_0 + a_1t_1 + a_2t_1^2 + a_3t_1^3 + a_4t_1^4 = 0,$$

which is precisely to say that v_1 lies on the hyperplane H. Similarly, v_2 lies on H.

Let now w be any other vertex of the cyclic polytope and let s be its moment curve parameter. We have

$$\begin{aligned} p(s) &= (s - t_1)^2(s - t_2)^2 \\ &> 0. \end{aligned}$$

That is,

$$a_0 + a_1s + a_2s^2 + a_3s^3 + a_4s^4 > 0,$$

and so w lies on the positive side of H. Therefore, H supports an edge between v_1 and v_2, as desired. □

Problem 2

Proposition 2. *If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq$ vert(P) is a set of vertices of P, then $F = \bigvee_{j=1}^{k}\{\mathbf{v}_j\}$ if and only if $k^{-1}\sum_{j=1}^{k}\mathbf{v}_k \in$ relint(F), where $F \in L(P)$.*

Proof. Let x denote the convex combination $k^{-1}\sum_{j=1}^{k}\mathbf{v}_k$.

($\Rightarrow$) Since F is convex and contains $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, we know immediately that $x \in F$. It remains to show that x is not on the boundary of F. To see this, consider any supporting hyperplane H intersecting P precisely in a subface F' of F. Since F is the *smallest* face of P containing $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, at least one of the v_i lies off of H. Moreover, all such vertices lie in a common closed halfspace induced by H, as H supports P. Thus, x does not lie on H, and so does not lie on F'. As this holds for any subface of F, we conclude that x lies on no subface of F, and so lies in relint(F).

($\Leftarrow$) Let H_F be a supporting hyperplane intersecting P precisely in F. All vertices in the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ lie in a common closed halfspace induce by H_F, since H_F supports P. Reasoning as before, the fact that $x \in$ relint(F) implies that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is contained in F.

To conclude that $F = \bigvee_{j=1}^{k}\{\mathbf{v}_j\}$, it remains to show that no subface of F contains $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Suppose, for the purpose of contradiction, that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is contained in a subface F' of F. Let $H_{F'}$ be a supporting hyperplane intersecting P precisely in F'. Thus, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset F'$ is contained in $H_{F'}$, and so x is contained in $H_{F'}$, which is contrary to the fact that $x \in \operatorname{relint}(F)$. Hence, no subface of F contains the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, as desired. □

Problem 3

Let $(f_0^{(d)}, \ldots, f_d^{(d)})$ denote the f-vector of the d-dimensional hypercube $[0,1]^d$. Define

$$q(x, y) = \sum_{d=0}^{\infty} \sum_{j=0}^{d} f_j^{(d)} x^j y^d.$$

Proposition 3. *For $q(x, y)$ as defined above, we have $q(x, y) = 1/(1 - 2y - xy)$.*

Proof. We first obtain a closed form for $f_j^{(d)}$. A j-dimensional face of a d-dimensional hypercube is expressed using j "free" coordinates and $d - j$ "fixed" coordinated. The free coordinates range over all possible values for that coordinate (namely, 0 and 1). There are $\binom{d}{j}$ ways to select the free coordinates. The fixed coordinates are set to either 0 or 1, which can be accomplished in 2^{d-j} ways. Thus, $f_j^{(d)} = \binom{d}{j} 2^{d-j}$.

We have now

$$\begin{aligned}
q(x, y) &= \sum_{d=0}^{\infty} \sum_{j=0}^{d} f_j^{(d)} x^j y^d \\
&= \sum_{d=0}^{\infty} \sum_{j=0}^{d} \binom{d}{j} 2^{d-j} x^j y^d \\
&= \sum_{d=0}^{\infty} (2 + x)^d y^d && \text{(by the binomial theorem)} \\
&= \sum_{d=0}^{\infty} ((2 + x)y)^d \\
&= \frac{1}{1 - (2 + x)y} && \text{(sum of a geometric series)} \\
&= \frac{1}{1 - 2y - xy}.
\end{aligned}$$

□

Problem 4

Let $\Lambda \subset \mathbb{R}^n$ be a lattice, $S \subset \Lambda$ with $|S| < \infty$, $P = \operatorname{conv}(S)$, and $d = \dim(P)$. Define

$$L_P(t) = |\Lambda \cap tP|,$$

where, for any set $U \subset \mathbb{R}^n$, $tU = \{t\mathbf{u} | \mathbf{u} \in U\}$. It is a theorem of Ehrhart that there exist $a_0, \ldots, a_d \in \mathbb{Q}$ so that $L_P(t) = \sum_{j=0}^{d} a_j t^j$ for all $t \in \mathbb{Z}$.

Proposition 4. *Using the notation above,*

1. $a_0 = 1$;

2. $L_{[0,1]^n}(t) = (t+1)^n$, *where* $\Lambda = \mathbb{Z}^n$.

3. $a_d = vol_d(P)/\det(\Lambda)$ *(*vol_d *is the d-dimensional Lebesgue measure);*

Proof. For part 1, observe that $L_P(0) = a_0$ by Ehrhart's formula. At the same time

$$\begin{aligned} L_P(0) &= |\Lambda \cap (0 \cdot P)| \\ &= |\Lambda \cap \{0\}| \\ &= |\{0\}| \\ &= 1. \end{aligned}$$

For part 2, we proceed by induction on n. For $n = 1$, we have

$$\begin{aligned} L_{[0,1]}(t) &= |\Lambda \cap (t \cdot [0,1])| \\ &= |\Lambda \cap [0,t]| \\ &= |\{0,1,\ldots,t\}| && \text{(here, } \Lambda = \mathbb{Z}^n) \\ &= t+1. \end{aligned}$$

Suppose now the result holds for $n \leq k$. We have

$$\begin{aligned} L_{[0,1]^{k+1}}(t) &= |\Lambda \cap (t \cdot [0,1]^{k+1})| \\ &= |\Lambda \cap [0,t]^{k+1}| \\ &= |\Lambda \cap [0,t]^k| \cdot |\Lambda \cap [0,t]| \\ &= (t+1)^k \cdot (t+1) \\ &= (t+1)^{k+1}. \end{aligned}$$

For part 3, we begin as in the proof of Minkowski's theorem. That is, we define a linear bijection f such that $f(\Lambda) = \mathbb{Z}^d$. Since P is convex, we have $\text{vol}_d(P) = \det(\Lambda)\text{vol}_d(f(P))$. Letting P' denote $f(P)$, the problem of showing $a_d = \frac{\text{vol}_d(P)}{\det(\Lambda)}$ for any lattice Λ is equivalent to showing $a_d = \text{vol}(P')$ in the lattice $\mathbb{Z}^d$.

Observe now by Erhart's formula

$$\begin{aligned} a_d &= \lim_{t\to\infty} \frac{\sum_{j=0}^{d} a_j t^j}{t^d} \\ &= \lim_{t\to\infty} \frac{|\mathbb{Z}^d \cap tP'|}{t^d}. \end{aligned}$$

Thus, we wish to establish $\lim_{t\to\infty} \frac{|\mathbb{Z}^d \cap tP'|}{t^d} = \text{vol}(P')$.

For any $\delta > 0$, let consider the cube $[0,\delta)^d$, which has volume δ^d. We have

$$\begin{aligned} a_d &= \lim_{t\to\infty} \frac{|\mathbb{Z}^d \cap (t \cdot [0,\delta)^d)|}{t^d} \\ &= \lim_{t\to\infty} \frac{|\mathbb{Z}^d \cap [0,\delta t)^d|}{t^d} \\ &= \lim_{t\to\infty} \frac{\delta^d t^d}{t^d} \\ &= \delta^d, \end{aligned}$$

and so the claim holds for a single cube of arbitrary size.

Let $\mathcal{C}_\delta$ be a collection of cubes of common size δ that approximate P'. Since the cubes are disjoint, we have

$$\lim_{\delta \downarrow 0} \sum_{C \in \mathcal{C}_d} \mathrm{vol}_d(C) = \mathrm{vol}(P').$$

Given a particular set of approximating cubes $\mathcal{C}_\delta$, let P'_δ denote the set $\bigcup_{C \in \mathcal{C}_\delta} C$. Finally, we have

$$\begin{aligned}
a_d &= \lim_{t \to \infty} \frac{|\mathbb{Z}^d \cap tP'|}{t^d} \\
&= \lim_{t \to \infty} \frac{|\mathbb{Z}^d \cap t\left(\lim_{\delta \downarrow 0} P'_\delta\right)|}{t^d} \\
&= \lim_{\delta \downarrow 0} \lim_{t \to \infty} \frac{|\mathbb{Z}^d \cap tP'_\delta|}{t^d} \\
&= \lim_{\delta \downarrow 0} \lim_{t \to \infty} \sum_{C \in \mathcal{C}_\delta} \frac{|\mathbb{Z}^d \cap tC|}{t^d} \\
&= \lim_{\delta \downarrow 0} \sum_{C \in \mathcal{C}_\delta} \lim_{t \to \infty} \frac{|\mathbb{Z}^d \cap tC|}{t^d} \\
&= \lim_{\delta \downarrow 0} \sum_{C \in \mathcal{C}_\delta} \mathrm{vol}_d(C) \\
&= \lim_{\delta \downarrow 0} \mathrm{vol}_d(P'_\delta) \\
&= \mathrm{vol}_d(P').
\end{aligned}$$

□