On Negative Dependency Graphs in Spaces of Generalized Random Matchings

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1 Preliminaries

A negative dependency graph for events $A_1, \ldots, A_n$ in some ambient probability space is a simple graph on $[n]$ satisfying

$$\Pr \left( A_i \mid \bigwedge_{j \in S} \overline{A_j} \right) \leq \Pr (A_i)$$

for any index $i$ and any subset $S \subseteq \{ j \mid ij \notin E(G) \}$, provided the conditional probability $\Pr(A_i \mid \bigwedge_{j \in S} \overline{A_j})$ is well-defined (i.e. $\Pr(\bigwedge_{j \in S} \overline{A_j}) > 0$).

Lemma 1. (Lovász Local Lemma) Let $A_1, \ldots, A_n$ be events with a negative dependency graph $G$. If there exist numbers $x_1, \ldots, x_n \in [0, 1)$ such that

$$\Pr (A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j)$$

for all $i$, then

$$\Pr \left( \bigwedge_{i=1}^{n} \overline{A_i} \right) \geq \prod_{i=1}^{n} (1 - x_i).$$

In words, if the events of interest can be the vertices of a negative dependency graph satisfying the specified bounds, then there is a nonzero probability of avoiding all the events. The lemma is commonly used in non-constructive proofs of the existence of combinatorial structures satisfying some list of desired properties.
The lemma, as it is presented above, is actually a generalization of the original lemma of Lovász, which required the stronger property that non-adjacent events in the graph are mutually independent (in such a case the graph is instead called a dependency graph). There are few examples in the literature in which the events form a proper negative dependency graph (i.e. a negative dependency graph that is not also a dependency graph). The purpose of the present work is to establish that the space of perfect $k$-matchings of the complete graph and the space of perfect matchings of the complete multipartite graph induce a proper negative dependency graph in a natural way.

2 $k$-Matchings in the Complete Graph

2.1 Definitions

A $k$-edge in the complete graph is any collection of $k$ vertices. A $k$-matching is a collection of pairwise disjoint $k$-edges. Two $k$-matchings $M$ and $M'$ are said to conflict if there exist $k$-edges $e \in M$ and $e' \in M'$ such that $|e \cap e'| \notin \{0, k\}$.

Let $\Omega_N$ denote the probability space of perfect $k$-matchings of the complete graph on $N$ vertices (where we require that $k$ divides $N$) equipped with the uniform distribution. Given a matching $M = \{e_1, \ldots, e_k\}$, define the event $A_M = \{M' \in \Omega_N | e_i \in M' \text{ for all } 1 \leq i \leq k\}$. An event $A$ is said to be canonical if $A = A_M$ for some matching $M$.

2.2 Negative Dependency Graph

Theorem 2. Let $\mathcal{M}$ be a collection of $k$-matchings in $K_N$. The graph $G = G(\mathcal{M})$ described below is a negative dependency graph for the canonical events $\{A_M | M \in \mathcal{M}\}$:

- $V(G) = \mathcal{M}$
- $E(G) = \{M_1M_2 | M_1 \in \mathcal{M} \text{ and } M_2 \in \mathcal{M} \text{ are in conflict}\}$.

Proof. We will prove the theorem by induction on $N$. The base case $N = k$ is trivial. Throughout, we assume that the vertex set of $K_N$ is $[N]$. There is a canonical injection from $[N]$ into $[N + s]$, and consequently from $V(K_N)$
to \( V(K_{N+s}) \) and from \( E(K_N) \) to \( E(K_{N+s}) \). (Note that a perfect matching in \( K_N \) will not be perfect in \( K_{N+s} \) for \( s > 0 \).) To emphasize the difference in the size of the vertex set, we use \( A^N_M \) to denote the event induced by the \( k \)-matching \( M \) among the matching of an \( N \)-vertex complete graph.

**Lemma 3.** For any collection \( \mathcal{M} \) of \( k \)-matchings in \( K_N \), we have

\[
\Pr \left( \bigwedge_{M \in \mathcal{M}} A^N_M \right) \leq \Pr \left( \bigwedge_{M \in \mathcal{M}} A^{N+k}_M \right).
\]

**Proof.** Let \( \mathcal{S} = \{S \mid S \subseteq [N + k - 1], |S| = k\} \). We partition the space of \( \Omega_{N+k} \) into \( \binom{N+k-1}{k-1} \) sets as follows: for each \( S \in \mathcal{S} \), let \( \mathcal{C}_S \) be the set of perfect matchings containing the \( k \)-edge \( S \cup \{N+k\} \). We have

\[
\Pr \left( \bigwedge_{M \in \mathcal{M}} A^{N+k}_M \right) = \sum_{S \in \mathcal{S}} \Pr \left( \bigwedge_{M \in \mathcal{M}} A^{N+k}_M \land \mathcal{C}_S \right).
\]

We observe that \( \mathcal{C}_S \subseteq \overline{A^{N+k}_M} \) if and only if \( M \) conflicts \( S \cup \{N+k\} \), a one-edge matching. Let \( \mathcal{B}_S \) be the subset of \( \mathcal{M} \) whose elements are not in conflict with the \( k \)-edge \( S \cup \{N+k\} \). (In particular, \( \mathcal{B}_{\{N+1,\ldots,N+k-1\}} = \mathcal{M} \).) We have

\[
\bigwedge_{M \in \mathcal{M}} A^{N+k}_M \land \mathcal{C}_S = \bigwedge_{M \in \mathcal{B}_S} A^{N+k}_M \land \mathcal{C}_S.
\]

Let \( \phi_S \) be any bijection between \( S \) and \( \{N+1,\ldots,N+k-1\} \). Note that \( \phi_S \) stabilizes \( \mathcal{B}_S \), interchanges \( \mathcal{C}_S \) and \( \mathcal{C}_{\{N+1,\ldots,N+k-1\}} \), and maps \( \bigwedge_{M \in \mathcal{B}_S} A^{N+k}_M \land \mathcal{C}_S \) to \( \bigwedge_{M \in \mathcal{B}_S} A^{N+k}_M \land \mathcal{C}_{\{N+1,\ldots,N+k-1\}} \). We have

\[
\Pr \left( \bigwedge_{M \in \mathcal{M}} A^{N+k}_M \right) = \sum_{S \in \mathcal{S}} \Pr \left( \bigwedge_{M \in \mathcal{M}} A^{N+k}_M \land \mathcal{C}_S \right)
\]

\[
= \sum_{S \in \mathcal{S}} \Pr \left( \bigwedge_{M \in \mathcal{M}} A^{N+k}_M \land \mathcal{C}_S \right)
\]

\[
= \sum_{S \in \mathcal{S}} \Pr \left( \bigwedge_{M \in \mathcal{B}_S} A^{N+k}_M \land \mathcal{C}_{\{N+1,\ldots,N+k-1\}} \right)
\]

\[
= \sum_{S \in \mathcal{S}} \Pr \left( \bigwedge_{M \in \mathcal{B}_S} A^{N+k}_M \mid \mathcal{C}_{\{N+1,\ldots,N+k-1\}} \right) \Pr \left( \mathcal{C}_{\{N+1,\ldots,N+k-1\}} \right)
\]

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\[
\sum_{S \in \mathcal{S}} \Pr \left( \bigwedge_{M \in \mathcal{B}_S} A_M^N \right)
\]
and estimating further
\[
\geq \frac{1}{\binom{N+k-1}{k-1}} \left( \binom{N+k-1}{k-1} \Pr \left( \bigwedge_{M \in \mathcal{M}} A_M^N \right) \right)
\]
\[
= \Pr \left( \bigwedge_{M \in \mathcal{M}} A_M^N \right).
\]

We return now to the proof of the theorem. For any fixed matching \(M \in \mathcal{M}\) and a subset \(\mathcal{J} \subseteq \mathcal{M}\) satisfying that for every \(M' \in \mathcal{J}\), \(M'\) is not in conflict with \(M\), it suffices to show that

\[
\Pr \left( \bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}} \mid A_M \right) \leq \Pr \left( \bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}} \right).
\]

Let \(\mathcal{J}' = \{M' \setminus M \mid M' \in \mathcal{J}\}\). Assume first that \(\phi \notin \mathcal{J}'\). Since every matching \(M'\) in \(\mathcal{J}\) is not in conflict with \(M\), the vertex set \(V(M' \setminus M)\) of \(M' \setminus M\) is disjoint from the vertex set \(V(M)\) of \(M\). Let \(T = V(M)\) be the set of vertices covered by the matching \(M\) and \(U\) be the set of vertices covered by at least one matching \(F \in \mathcal{J}'\). We have \(T \cap U = \emptyset\). Let \(\pi\) be a permutation of \([N]\) mapping \(T\) to \([N-|T|+1, N-|T|+2, \ldots, N]\). We have \(\pi(T) \cap \pi(U) = \emptyset\). Thus, \(\pi(U) \subseteq [N-|T|]\). Let \(\pi(\mathcal{J}') = \{\pi(F) \mid F \in \mathcal{J}'\}\) and \(F' = \pi(F)\). We obtain

\[
\Pr \left( \bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}} \mid A_M \right) = \frac{\Pr \left( \bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}} \wedge A_M \right)}{\Pr (A_M)}
\]
\[
= \frac{\Pr \left( \bigwedge_{M' \in \mathcal{J}} \overline{A_{M' \setminus M}} \wedge A_M \right)}{\Pr (A_M)}
\]
\[
= \frac{\Pr \left( \bigwedge_{F \in \mathcal{J}'} \overline{A_F} \wedge A_M \right)}{\Pr (A_M)}
\]
\[
= \Pr \left( \bigwedge_{F \in \mathcal{J}'} \overline{A_F} \mid A_M \right)
\]

\(\square\)
\[
\begin{align*}
&= \Pr \left( \bigwedge_{F' \in \pi(J')} A_{F'}^N \mid A_{\pi(M)} \right) \\
&= \Pr \left( \bigwedge_{F' \in \pi(J')} A_{F'}^{N-|F'|} \right) \\
&\leq \Pr \left( \bigwedge_{F' \in \pi(J')} A_{F'}^N \right) \quad \text{(by lemma)} \\
&= \Pr \left( \bigwedge_{F \in J'} A_F^N \right) \\
&= \Pr \left( \bigwedge_{M' \not\in J \cap \pi(M)} A_{M' \setminus M}^N \right) \\
&\leq \Pr \left( \bigwedge_{M' \in J} A_{M'}^N \right).
\end{align*}
\]

If \( \emptyset \in J' \), then the LHS of the estimate above is zero, and therefore we have nothing to do.

\[\square\]

3 Matchings in the Complete Multipartite Graph

3.1 Definitions

Let \( V_1, \ldots, V_m \) be sets indexed such that \( V_1 \) is of least cardinality among the \( V_i \). A matching of these sets is a tuple

\[(U_1, \ldots, U_m, f_2, \ldots, f_m)\]

satisfying

- \( U_i \subseteq V_i \) for each \( 1 \leq i \leq m \), and \\
- \( f_i \) is a bijection from \( U_1 \) to \( U_i \) for each \( 2 \leq i \leq m \).

Denote by \( \mathcal{M}(V_1, \ldots, V_m) \) the collection of all matchings of the sets \( V_1, \ldots, V_m \) (we write simply \( \mathcal{M} \) when the underlying sets are understood). The collection
of saturated matchings (i.e. those matchings satisfying $U_1 = V_1$) will be denoted $\mathcal{I}(V_1, \ldots, V_m)$ (or simply $\mathcal{I}$).

For the remainder of the discussion, let $M = (U_1, \ldots, U_m, f_2, \ldots, f_m)$ and $M' = (U'_1, \ldots, U'_m, f'_2, f'_m)$ be two arbitrary matchings of $V_1, \ldots, V_m$.

The matchings $M$ and $M'$ are said to conflict each other there exists $u \in U_1$ and $u' \in U'_1$ such that

$$|\{f_i(u) \mid i \in [m]\} \cap |\{f'_i(u') \mid i \in [m]\}| \notin \{0, m\}.$$

Loosely, two matchings conflict if their union (after supressing repeat mappings) is not again a matching.

Define the event $A_M \subseteq \mathcal{I}$ (where we endow $\mathcal{I}$ with the uniform probability measure) as

$$A_M = \{ (U''_1, \ldots, U''_m, f''_2, \ldots, f''_m) \in \mathcal{I} \mid \text{for each } i, f''_i(u) = f_i(u) \forall u \in U_1 \}.$$

An event $A \subseteq \mathcal{I}$ is said to be canonical if $A = A_M$ for some matching $M$. Two canonical matchings conflict each other if their associated matchings conflict. Note that if two events conflict each other, then they are disjoint.

### 3.2 Negative Dependency Graph

We establish a sufficient condition for negative dependency graphs for the space $\mathcal{I}$ endowed with the uniform probability measure by showing the following theorem.

**Theorem 4.** Let $A_1, \ldots, A_n$ be canonical events in $\mathcal{I}$. The graph $G$ on $[n]$ with

$$E(G) = \{ ij \mid A_i \text{ and } A_j \text{ conflict} \}$$

is a negative dependency graph for the events $A_1, \ldots, A_n$.

**Proof.** We are supposed to show the inequality

$$\Pr \left( A_i \mid \bigwedge_{j \in J} \overline{A_j} \right) \leq \Pr(A_i)$$

for any index $i$, where $J \subseteq \{ j \mid A_i \text{ and } A_j \text{ do not conflict} \}$. If $\Pr(\bigwedge_{j \in J} \overline{A_j}) = 0$, then there is nothing to prove. Hence, we assume $\Pr(\bigwedge_{j \in J} \overline{A_j}) > 0$. Under
this assumption, it is equivalent to show
\[
\Pr \left( \bigwedge_{j \in J} \overline{A}_j \mid A_i \right) \leq \Pr \left( \bigwedge_{j \in J} \overline{A}_j \right).
\]

For \(1 \leq k \leq n\), let \(M_k = (U_1^k, \ldots, U_m^k, f_2^k, \ldots, f_m^k)\) be the corresponding matching of the event \(A_k\).

**Claim:** For any matching \(M = (U_1^i, U_2^i, \ldots, U_m^i, f_2^i, \ldots, f_m^i)\) and any index \(i\),
\[
\Pr (A_M) = \Pr (A_i).
\] (1)

Moreover, if \(J \subseteq \{ j \mid A_i \text{ and } A_j \text{ do not conflict} \}\), then
\[
\Pr \left( \left( \bigwedge_{j \in J} \overline{A}_j \right) A_M \right) \geq \Pr \left( \left( \bigwedge_{j \in J} \overline{A}_j \right) A_i \right).
\] (2)

**Proof of Claim:** Fix a matching \(M\) as above. Let \(J'\) be the set of indices \(j \in J\) such that \(A_j\) does not conflict with \(A_M\). Clearly,
\[
\left( \bigwedge_{j \in J} \overline{A}_j \right) A_M = \left( \bigwedge_{j \in J'} \overline{A}_j \right) \left( \bigwedge_{j \in J \setminus J'} \overline{A}_j \right) A_M.
\]

If \(j \in J \setminus J'\), then \(A_j\) conflicts with \(A_M\), and so \(A_M \subseteq \overline{A}_j\). Therefore,
\[
\overline{A}_j A_M = A_M.
\]

Thus, whether \(J \setminus J'\) is empty or not, we have
\[
\left( \bigwedge_{j \in J \setminus J'} \overline{A}_j \right) A_M = A_M,
\]
from which it follows that
\[
\left( \bigwedge_{j \in J} \overline{A}_j \right) A_M = \left( \bigwedge_{j \in J'} \overline{A}_j \right) A_M.
\] (3)

Define now a **permutation system** to be a collection of permutations
\[
\{ \rho_i \mid 2 \leq i \leq m \},
\]
where each $\rho_i$ is a permutation on $V_i$. Given a permutation system $P$, define the mapping $\pi_P : \mathcal{M} \to \mathcal{M}$ where $\pi_P(M) = M'$ means

$$f_i'(u) = \rho_i(f_i(u))$$

for all $u \in U_1$. Observe that $U_1 = U_1'$, and, if every $U_i$ consists of fixpoints of $\rho_i$, then $M = M'$.

Note that for any permutation system $P$, if $\pi_P(M) = M'$, then $\pi_P(A_M) = \pi_P(A_{M'})$.

Let $P$ be a permutation system $\{\rho_k : V_k \to V_k | 2 \leq k \leq m\}$ where each $\rho_k$ satisfies

- $\rho_k(u) = u$ for any $u \in \bigcup_{j \in J'} U_k^j$, and
- $\rho_k(u) = f_i^k(f_k^-(u))$ for any $u \in U_k$.

By the definition of $J'$, we have that, for each $j \in J'$, if $u \in U_1^j \cap U_1^j$, then $f_i^k(u) = f_j^k(u) = f_k(u)$. Therefore, such a $\rho_k$ exists for each $2 \leq k \leq m$. Moreover, for each $j \in J'$, $U_k^j$ consists of fixpoints of $\rho_k$, $\rho_k(U_k) = U_k^j$, and for $u \in U_1^j$, $\rho_k(f_k(u)) = f_i^k(u)$ for each $2 \leq k \leq m$.

The above implies that $\pi_P(M_j) = M_j$, from which (1) follows. Also, for each $j \in J'$, we have $\pi_P(M_j) = M_j$. Thus, for each $j \in J'$,

$$\pi_P(A_j A_M) = A_j A_M,$$

and so

$$\pi_P \left( \bigwedge_{j \in J'} A_j \right) A_M = \left( \bigwedge_{j \in J'} A_j \right) A_M. \quad (4)$$

Using equations (3) and (4), we obtain

$$\Pr \left( \bigwedge_{j \in J} A_j \right) A_M = \Pr \left( \bigwedge_{j \in J'} A_j \right) A_M = \Pr \left( \bigwedge_{j \in J'} A_j \right) A_i \geq \Pr \left( \bigwedge_{j \in J} A_j \right) A_i,$$

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thus completing the proof of the claim.

For the fixed set $U_1$, let $\mathcal{M}'$ denote the collection of matchings with $U_1 = U_1$. The collection of events

$$\{A_M \mid M \in \mathcal{M}'\}$$

forms a partition of the space $\mathcal{I}$.

From this partition and equations (1) and (2), we get

$$\Pr\left(\bigwedge_{j \in J} \overline{A_j}\right) = \sum_{M \in \mathcal{M}'} \Pr\left(\left(\bigwedge_{j \in J} \overline{A_j}\right)A_M\right)$$

$$\geq \sum_{M \in \mathcal{M}'} \Pr\left(\left(\bigwedge_{j \in J} \overline{A_j}\right)A_i\right)$$

$$= \sum_{M \in \mathcal{M}'} \Pr\left(\bigwedge_{j \in J} \overline{A_j} \mid A_i\right) \Pr(A_i)$$

$$= \sum_{M \in \mathcal{M}'} \Pr\left(\bigwedge_{j \in J} \overline{A_j} \mid A_i\right) \Pr(A_M)$$

$$= \Pr\left(\bigwedge_{j \in J} \overline{A_j} \mid A_i\right).$$

$\square$