On Negative Dependency Graphs in Spaces of Generalized Random Matchings

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1 Preliminaries

A *negative dependency graph* for events $A_1, \ldots, A_n$ in some ambient probability space is a simple graph on $[n]$ satisfying

$$\Pr\left(A_i \mid \bigwedge_{j \in S} \overline{A_j}\right) \leq \Pr\left(A_i\right)$$

for any index i and any subset $S \subseteq \{j \mid ij \notin E(G)\}$, provided the conditional probability $\Pr(A_i \mid \bigwedge_{j \in S} \overline{A_j})$ is well-defined (i.e. $\Pr(\bigwedge_{j \in S} \overline{A_j}) > 0$).

Lemma 1. *(Lovász Local Lemma) Let $A_1, \ldots, A_n$ be events with a negative dependency graph G. If there exist numbers $x_1, \ldots, x_n \in [0, 1)$ such that*

$$\Pr\left(A_i\right) \leq x_i \prod_{ij \in E(G)} (1 - x_j)$$

for all i, then

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n} (1 - x_i).$$

In words, if the events of interest can be the vertices of a negative dependency graph satisfying the specified bounds, then there is a nonzero probability of avoiding all the events. The lemma is commonly used in nonconstructive proofs of the existence of combinatorial structures satisfying some list of desired properties.

The lemma, as it is presented above, is actually a generalization of the original lemma of Lovász, which required the stronger property that non-adjacent events in the graph are mutually independent (in such a case the graph is instead called a dependency graph). There are few examples in the literature in which the events form a proper negative dependency graph (i.e. a negative dependency graph that is not also a dependency graph). The purpose of the present work is to establish that the space of perfect k-matchings of the complete graph and the space of perfect matchings of the complete multipartite graph induce a proper negative dependency graph in a natural way.

2 k-Matchings in the Complete Graph

2.1 Definitions

A *k-edge* in the complete graph is any collection of k vertices. A *k-matching* is a collection of pairwise disjoint k-edges. Two k-matchings M and M' are said to *conflict* if there exist k-edges $e \in M$ and $e' \in M'$ such that $|e \cap e'| \notin \{0, k\}$.

Let Ω_N denote the probability space of perfect k-matchings of the complete graph on N vertices (where we require that k divides N) equipped with the uniform distribution. Given a matching $M = \{e_1, \dots, e_k\}$, define the event $A_M = \{M' \in \Omega_N \mid e_i \in M' \text{ for all } 1 \leq i \leq k\}$. An event A is said to be *canonical* if $A = A_M$ for some matching M.

2.2 Negative Dependency Graph

Theorem 2. *Let $\mathcal{M}$ be a collection of k-matchings in K_N. The graph $G = G(\mathcal{M})$ described below is a negative dependency graph for the canonical events $\{A_M \mid M \in \mathcal{M}\}$:*

- $V(G) = \mathcal{M}$
- $E(G) = \{M_1M_2 \mid M_1 \in \mathcal{M} \text{ and } M_2 \in \mathcal{M} \text{ are in conflict}\}$.

Proof. We will prove the theorem by induction on N. The base case $N = k$ is trivial. Throughout, we assume that the vertex set of K_N is $[N]$. There is a canonical injection from $[N]$ into $[N + s]$, and consequently from $V(K_N)$

to $V(K_{N+s})$ and from $E(K_N)$ to $E(K_{N+s})$. (Note that a perfect matching in K_N will not be perfect in K_{N+s} for $s > 0$.) To emphasize the difference in the size of the vertex set, we use A_M^N to denote the event induced by the k-matching M among the matching of an N-vertex complete graph.

Lemma 3. *For any collection $\mathcal{M}$ of k-matchings in K_N, we have*

$$\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_M^N}\right) \le \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_M^{N+k}}\right).$$

Proof. Let $\mathcal{S} = \{S \mid S \subseteq [N+k-1], |S| = k\}$. We partition the space of Ω_{N+k} into $\binom{N+k-1}{k-1}$ sets as follows: for each $S \in \mathcal{S}$, let $\mathcal{C}_S$ be the set of perfect matchings containing the k-edge $S \cup \{N+k\}$. We have

$$\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_M^{N+k}}\right) = \sum_{S\in\mathcal{S}} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_M^{N+k}} \wedge \mathcal{C}_S\right).$$

We observe that $\mathcal{C}_S \subseteq \overline{A_M^{N+k}}$ if and only if M conflicts $S \cup \{N+k\}$, a one-edge matching. Let $\mathcal{B}_S$ be the subset of $\mathcal{M}$ whose elements are not in conflict with the k-edge $S \cup \{N+k\}$. (In particular, $\mathcal{B}_{\{N+1,\ldots,N+k-1\}} = \mathcal{M}$.) We have

$$\bigwedge_{M\in\mathcal{M}} \overline{A_M^{N+k}} \wedge \mathcal{C}_S = \bigwedge_{M\in\mathcal{B}_S} \overline{A_M^{N+k}} \wedge \mathcal{C}_S.$$

Let ϕ_S be any bijection between S and $\{N+1,\ldots,N+k-1\}$. Note that ϕ_S stabilizes $\mathcal{B}_S$, interchanges $\mathcal{C}_S$ and $\mathcal{C}_{\{N+1,\ldots,N+k-1\}}$, and maps $\bigwedge_{M\in\mathcal{B}_S} \overline{A_M^{N+k}} \wedge \mathcal{C}_S$ to $\bigwedge_{M\in\mathcal{B}_S} \overline{A_M^{N+k}} \wedge \mathcal{C}_{\{N+1,\ldots,N+k-1\}}$. We have

$$\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_M^{N+k}}\right) &= \sum_{S\in\mathcal{S}} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_M^{N+k}} \wedge \mathcal{C}_S\right) \\
&= \sum_{S\in\mathcal{S}} \Pr\left(\bigwedge_{M\in\mathcal{B}_S} \overline{A_M^{N+k}} \wedge \mathcal{C}_S\right) \\
&= \sum_{S\in\mathcal{S}} \Pr\left(\bigwedge_{M\in\mathcal{B}_S} \overline{A_M^{N+k}} \wedge \mathcal{C}_{\{N+1,\ldots,N+k-1\}}\right) \\
&= \sum_{S\in\mathcal{S}} \Pr\left(\bigwedge_{M\in\mathcal{B}_S} \overline{A_M^{N+k}} \mid \mathcal{C}_{\{N+1,\ldots,N+k-1\}}\right) \Pr\left(\mathcal{C}_{\{N+1,\ldots,N+k-1\}}\right)
\end{aligned}$$

$$= \frac{1}{\binom{N+k-1}{k-1}} \sum_{S \in \mathcal{S}} \Pr\left(\bigwedge_{M \in \mathcal{B}_S} \overline{A_M^N} \right)$$

and estimating further

$$\geq \frac{1}{\binom{N+k-1}{k-1}} \left(\binom{N+k-1}{k-1} \Pr\left(\bigwedge_{M \in \mathcal{M}} \overline{A_M^N} \right) \right)$$
$$= \Pr\left(\bigwedge_{M \in \mathcal{M}} \overline{A_M^N} \right).$$

□

We return now to the proof of the theorem. For any fixed matching $M \in \mathcal{M}$ and a subset $\mathcal{J} \subseteq \mathcal{M}$ satisfying that for every $M' \in \mathcal{J}$, M' is not in conflict with M, it suffices to show that

$$\Pr\left(\bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}} \mid A_M \right) \leq \Pr\left(\bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}} \right).$$

Let $\mathcal{J}' = \{M' \setminus M \mid M' \in \mathcal{J}\}$. Assume first that $\phi \notin \mathcal{J}'$. Since every matching M' in $\mathcal{J}$ is not in conflict with M, the vertex set $V(M' \setminus M)$ of $M' \setminus M$ is disjoint from the vertex set $V(M)$ of M. Let $T = V(M)$ be the set of vertices covered by the matching M and U be the set of vertices covered by at least one matching $F \in \mathcal{J}'$. We have $T \cap U = \emptyset$. Let π be a permutation of $[N]$ mapping T to $\{N-|T|+1, N-|T|+2, \ldots, N\}$. We have $\pi(T) \cap \pi(U) = \emptyset$. Thus, $\pi(U) \subseteq [N-|T|]$. Let $\pi(\mathcal{J}') = \{\pi(F) \mid F \in \mathcal{J}'\}$ and $F' = \pi(F)$. We obtain

$$\Pr\left(\bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}} \mid A_M \right) = \frac{\Pr\left(\bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}} \wedge A_M \right)}{\Pr\left(A_M\right)}$$
$$= \frac{\Pr\left(\bigwedge_{M' \in \mathcal{J}} \overline{A_{M' \setminus M}} \wedge A_M \right)}{\Pr\left(A_M\right)}$$
$$= \frac{\Pr\left(\bigwedge_{F \in \mathcal{J}'} \overline{A_F} \wedge A_M \right)}{\Pr\left(A_M\right)}$$
$$= \Pr\left(\bigwedge_{F \in \mathcal{J}'} \overline{A_F} \mid A_M \right)$$

$$
\begin{aligned}
&= \Pr\left(\bigwedge_{F' \in \pi(\mathcal{J}')} \overline{A_{F'}^{N}} \mid A_{\pi(M)} \right) \\
&= \Pr\left(\bigwedge_{F' \in \pi(\mathcal{J}')} \overline{A_{F'}^{N-|T|}} \right) \\
&\leq \Pr\left(\bigwedge_{F' \in \pi(\mathcal{J}')} \overline{A_{F'}^{N}} \right) && \text{(by lemma)} \\
&= \Pr\left(\bigwedge_{F \in \mathcal{J}'} \overline{A_{F}^{N}} \right) \\
&= \Pr\left(\bigwedge_{M' \in \mathcal{J}} \overline{A_{M' \setminus M}^{N}} \right) \\
&\leq \Pr\left(\bigwedge_{M' \in \mathcal{J}} \overline{A_{M'}^{N}} \right).
\end{aligned}
$$

If $\emptyset \in \mathcal{J}'$, then the LHS of the estimate above is zero, and therefore we have nothing to do. $\square$

3 Matchings in the Complete Multipartite Graph

3.1 Definitions

Let $V_1, \dots, V_m$ be sets indexed such that V_1 is of least cardinality among the V_i. A *matching* of these sets is a tuple

$$(U_1, \dots, U_m, f_2, \dots, f_m)$$

satisfying

- $U_i \subseteq V_i$ for each $1 \leq i \leq m$, and
- f_i is a bijection from U_1 to U_i for each $2 \leq i \leq m$.

Denote by $\mathcal{M}(V_1, \dots, V_m)$ the collection of all matchings of the sets $V_1, \dots, V_m$ (we write simply $\mathcal{M}$ when the underlying sets are understood). The collection of saturated matchings (i.e. those matchings satisfying $U_1 = V_1$) will be denoted $\mathcal{I}(V_1, \ldots, V_m)$ (or simply $\mathcal{I}$).

For the remainder of the discussion, let $M = (U_1, \ldots, U_m, f_2, \ldots, f_m)$ and $M' = (U'_1, \ldots, U'_m, f'_2, f'_m)$ be two arbitrary matchings of $V_1, \ldots, V_m$.

The matchings M and M' are said to *conflict* each other there exists $u \in U_1$ and $u' \in U'_1$ such that

$$|\{f_i(u) \mid i \in [m]\}| \cap |\{f'_i(u') \mid i \in [m]\}| \notin \{0, m\}.$$

Loosely, two matchings conflict if their union (after supressing repeat mappings) is not again a matching.

Define the event $A_M \subset \mathcal{I}$ (where we endow $\mathcal{I}$ with the uniform probability measure) as

$$A_M = \{(U''_1, \ldots, U''_m, f''_2, \ldots, f''_m) \in \mathcal{I} \mid \text{for each } i,\ f''_i(u) = f_i(u)\, \forall u \in U_1\}.$$

An event $A \subseteq \mathcal{I}$ is said to be *canonical* if $A = A_M$ for some matching M. Two canonical matchings conflict each other if their associated matchings conflict. Note that if two events conflict each other, then they are disjoint.

3.2 Negative Dependency Graph

We establish a sufficient condition for negative dependency graphs for the space $\mathcal{I}$ endowed with the uniform probability measure by showing the following theorem.

Theorem 4. *Let $A_1 \ldots, A_n$ be canonical events in $\mathcal{I}$. The graph G on $[n]$ with*

$$E(G) = \{ij \mid A_i \text{ and } A_j \text{ conflict}\}$$

is a negative dependency graph for the events $A_1, \ldots, A_n$.

Proof. We are supposed to show the inequality

$$\Pr\left(A_i \mid \bigwedge_{j \in J} \overline{A_j}\right) \leq \Pr(A_i)$$

for any index i, where $J \subseteq \{j \mid A_i \text{ and } A_j \text{ do not conflict}\}$. If $\Pr\left(\wedge_{j\in J}\overline{A_j}\right) = 0$, then there is nothing to prove. Hence, we assume $\Pr\left(\wedge_{j\in J}\overline{A_j}\right) > 0$. Under this assumption, it is equivalent to show

$$\Pr\left(\bigwedge_{j\in J}\overline{A_j}\mid A_i\right)\leq\Pr\left(\bigwedge_{j\in J}\overline{A_j}\right).$$

For $1\leq k\leq n$, let $M_k=(U_1^k,\ldots,U_m^k,f_2^k,\ldots,f_m^k)$ be the corresponding matching of the event A_k.
Claim: For any matching $M=(U_1^i,U_2\ldots,U_m,f_2,\ldots,f_m)$ and any index i,

$$\Pr\left(A_M\right)=\Pr\left(A_i\right). \tag{1}$$

Moreover, if $J\subseteq\{j\mid A_i \text{ and } A_j \text{ do not conflict}\}$, then

$$\Pr\left(\left(\bigwedge_{j\in J}\overline{A_j}\right)A_M\right)\geq\Pr\left(\left(\bigwedge_{j\in J}\overline{A_j}\right)A_i\right). \tag{2}$$

Proof of Claim: Fix a matching M as above. Let J' be the set of indices $j\in J$ such that A_j does not conflict with A_M. Clearly,

$$\left(\bigwedge_{j\in J}\overline{A_j}\right)A_M=\left(\bigwedge_{j\in J'}\overline{A_j}\right)\left(\bigwedge_{j\in J\setminus J'}\overline{A_j}\right)A_M.$$

If $j\in J\setminus J'$, then A_j conflicts with A_M, and so $A_M\subseteq\overline{A_j}$. Therefore,

$$\overline{A_j}A_M=A_M.$$

Thus, whether $J\setminus J'$ is empty or not, we have

$$\left(\bigwedge_{j\in J\setminus J'}\overline{A_j}\right)A_M=A_M,$$

from which it follows that

$$\left(\bigwedge_{j\in J}\overline{A_j}\right)A_M=\left(\bigwedge_{j\in J'}\overline{A_j}\right)A_M. \tag{3}$$

Define now a *permutation system* to be a collection of permutations

$$\{\rho_i\mid 2\leq i\leq m\},$$

where each ρ_i is a permutation on V_i. Given a permutation system P, define the mapping $\pi_P : \mathcal{M} \to \mathcal{M}$ where $\pi_P(M) = M'$ means

$$f'_i(u) = \rho_i(f_i(u))$$

for all $u \in U_1$. Observe that $U_1 = U'_1$, and, if every U_i consists of fixpoints of ρ_i, then $M = M'$.

Note that for any permutation system P, if $\pi_P(M) = M'$, then $\pi_P(A_M) = \pi_P(A_{M'})$.

Let P be a permutation system $\{\rho_k : V_k \to V_k \mid 2 \le k \le m\}$ where each ρ_k satisfies

- $\rho_k(u) = u$ for any $u \in \bigcup_{j \in J'} U_k^j$, and
- $\rho_k(u) = f_k^i(f_k^{\leftarrow}(u))$ for any $u \in U_k$.

By the definition of J', we have that, for each $j \in J'$, if $u \in U_1^i \cap U_1^j$, then $f_k^i(u) = f_k^j(u) = f_k(u)$. Therefore, such a ρ_k exists for each $2 \le k \le m$. Moreover, for each $j \in J'$, U_k^j consists of fixpoints of ρ_k, $\rho_k(U_k) = U_k^i$, and for $u \in U_1^i$, $\rho_k(f_k(u)) = f_k^i(u)$ for each $2 \le k \le m$.

The above implies that $\pi_P(M) = M_i$, from which (1) follows. Also, for each $j \in J'$, we have $\pi_P(M_j) = M_j$. Thus, for each $j \in J'$,

$$\pi_P(\overline{A_j} A_M) = \overline{A_j} A_M,$$

and so

$$\pi_P\left(\left(\bigwedge_{j \in J'} \overline{A_j}\right) A_M\right) = \left(\bigwedge_{j \in J'} \overline{A_j}\right) A_M. \tag{4}$$

Using equations (3) and (4), we obtain

$$\begin{aligned}
\Pr\left(\left(\bigwedge_{j \in J} \overline{A_j}\right) A_M\right) &= \Pr\left(\left(\bigwedge_{j \in J'} \overline{A_j}\right) A_M\right) \\
&= \Pr\left(\left(\bigwedge_{j \in J'} \overline{A_j}\right) A_i\right) \\
&\ge \Pr\left(\left(\bigwedge_{j \in J} \overline{A_j}\right) A_i\right),
\end{aligned}$$

thus completing the proof of the claim.

For the fixed set U_1^i, let $\mathcal{M}'$ denote the collection of matchings with $U_1 = U_1^i$. The collection of events

$$\{A_M \mid M \in \mathcal{M}'\}$$

forms a partition of the space $\mathcal{I}$.

From this partition and equations (1) and (2), we get

$$\begin{aligned}
\Pr\left(\bigwedge_{j\in J} \overline{A_j}\right) &= \sum_{M\in\mathcal{M}'} \Pr\left(\left(\bigwedge_{j\in J} \overline{A_j}\right) A_M\right) \\
&\geq \sum_{M\in\mathcal{M}'} \Pr\left(\left(\bigwedge_{j\in J} \overline{A_j}\right) A_i\right) \\
&= \sum_{M\in\mathcal{M}'} \Pr\left(\bigwedge_{j\in J} \overline{A_j} \mid A_i\right) \Pr\left(A_i\right) \\
&= \sum_{M\in\mathcal{M}'} \Pr\left(\bigwedge_{j\in J} \overline{A_j} \mid A_i\right) \Pr\left(A_M\right) \\
&= \Pr\left(\bigwedge_{j\in J} \overline{A_j} \mid A_i\right).
\end{aligned}$$

□