Checking Hats with Local Lemmata

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SIUC Mathematics Conference
May 17, 2016

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Austin Mohr
told you something about the
probability of an intersection.

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right)$$

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Some Context

Introductory Probability

For **independent events**, we have

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \Pr\left(A_i\right).$$

Some Context

Lovász Local Lemma (Erdős, Lovász, Spencer)

You can get a **lower bound** on

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right)$$

provided **most events are independent**.

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Lopsided Lovász Local Lemma (Erdős, Spencer)

You can get a **lower bound** on

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right)$$

without any independence.

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Some Context

Positive Dependence Local Lemma (L. Lu, L. A. Székely)

You can get an **upper bound** on

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right)$$

without any independence.

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Some Context

In the right situations, using both

- Lopsided Lovász Local Lemma
- Positive Dependence Local Lemma

together give lower and upper bounds that are **asymptotically equal**

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Some Context

In the right situations, using both

- Lopsided Lovász Local Lemma
- Positive Dependence Local Lemma

together give lower and upper bounds that are **asymptotically equal**, and you don't have to be very good at probability to do this. (Mohr)

Hat-Check Problem

- n customers check hats at a restaurant.
- Hats returned randomly.
- What is the probability that no customer receives their own hat?

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Hat-Check Problem

- n customers check hats at a restaurant.
- Hats returned randomly.
- What is the probability that no customer receives their own hat?

Let H_i be the set of arrangements where customer i receives their own hat.
We are interested in

$$\Pr\left(\bigcap_{i=1}^{n} \overline{H_i}\right).$$

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Independence for Checking Hats

$$\Pr\left(H_j \cap H_k\right) = \frac{(n-2)!}{n!} = \frac{1}{n^2 - n}$$

Independence for Checking Hats

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while

$$\Pr(H_j)\Pr(H_k) = \frac{(n-1)!}{n!} \cdot \frac{(n-1)!}{n!} = \frac{1}{n^2}.$$

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Independence for Checking Hats

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No two events are independent,
so the classical local lemma cannot be applied.

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Lopsided Lovász Local Lemma

Lemma (**Lopsided** Lovász Local Lemma - Erdős, Spencer 1991)

Let $\{A_i \mid i \in [n]\}$ *be a collection of events having* **negative dependence graph** G.
If there are real numbers $x_i \in [0, 1)$ *such that*

$$\Pr(A_i) \le x_i \prod_{ij \in E(G)} (1 - x_j)$$

for all i, *then*

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \ge \prod_{i=1}^{n} (1 - x_i).$$

Negative dependence Graph

Negative dependence Graph G

- $V(G) = \{A_i \mid i \in [n]\}$ events
- Edges situated such that

$$\Pr\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) \leq \Pr\left(A_i\right)$$

where S is any collection of non-neighbors of A_i.

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Negative dependence Graph for Checking Hats

Theorem (Lu, Székely 2011; M 2016+)

The graph with vertex set $\{H_i \mid i \in [n]\}$ and no edges is a negative dependence graph.

For hat-check events,
the negative dependence condition always holds.

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Lopsided Lovász Local Lemma for Checking Hats

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Lopsided Lovász Local Lemma for Checking Hats

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$$\Pr(H_i) \leq x_i$$

for all i, then

$$\Pr\left(\bigcap_{i=1}^{n} \overline{H_i}\right) \geq \prod_{i=1}^{n}(1 - x_i).$$

Choose $x_i = \frac{1}{n}$ for all i, yielding

$$\Pr\left(\bigcap_{i=1}^{n} \overline{H_i}\right) \geq \left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}.$$

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Negative dependence Graph for Checking Hats

Proof of edgeless negative dependence graph.
We need to establish,

$$\Pr\left(H_n \,\middle|\, \bigcap_{j=1}^{k} \overline{H_j}\right) \leq \Pr\left(H_n\right),$$

for all k,

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$$\Pr\left(H_n \;\middle|\; \bigcap_{j=1}^{k} \overline{H_j}\right) \leq \Pr\left(H_n\right),$$

for all k, which is equivalent to

$$n\left|H_n \cap \bigcap_{j=1}^{k} \overline{H_j}\right| \leq \left|\bigcap_{j=1}^{k} \overline{H_j}\right|.$$

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Negative dependence Graph for Checking Hats

$$n \left| H_n \cap \bigcap_{j=1}^{k} \overline{H_j} \right| \leq \left| \bigcap_{j=1}^{k} \overline{H_j} \right|$$

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Given an arrangement in which

- customer n received their own hat and
- none of the first k customers received their own hat,

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Negative dependence Graph for Checking Hats

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Given an arrangement in which

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can we construct n distinct arrangements in which

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Given an arrangement in which

- customer n received their own hat and
- none of the first k customers received their own hat,

can we construct n distinct arrangements in which

- none of the first k customers received their own hat?

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Negative dependence Graph for Checking Hats

Customer	1	2	3	4
Hat	3	1	2	4

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Negative dependence Graph for Checking Hats

Customer	1	2	3	4
Hat	3	1	2	4
	4	1	2	3

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Negative dependence Graph for Checking Hats

Customer	**1**	**2**	**3**	**4**
Hat	3	1	2	4
	4	1	2	3
	3	4	2	1

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Customer	**1**	**2**	**3**	**4**
Hat	3	1	2	4
	4	1	2	3
	3	4	2	1
	3	1	4	2

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Negative dependence Graph for Checking Hats

Customer	**1**	**2**	**3**	**4**
Hat	3	1	2	4
	4	1	2	3
	3	4	2	1
	3	1	4	2

This establishes the existence of the edgeless negative dependence graph, so the lopsided Lovász local lemma gives the lower bound

$$\Pr\left(\bigcap_{i=1}^{n} \overline{H_i}\right) \geq \left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}.$$

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Positive Dependence Local Lemma

Theorem (Positive Dependence Local Lemma - Lu, Székely 2011)

*If $A_1, \ldots, A_n$ are events with an ϵ-***near positive dependency graph***, then*

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \leq \prod_{i=1}^{n} \left[1 - (1-\epsilon)\Pr(A_i)\right].$$

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ϵ-Near Positive Dependency Graph G

- $V(G) = \{A_i \mid i \in [n]\}$ events
- Edges situated such that

$$\Pr\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) \geq (1 - \epsilon) \Pr(A_i)$$

where S is any collection of non-neighbors of A_i and $\epsilon > 0$.

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ϵ-Near Positive Dependency Graph

ϵ-Near Positive Dependency Graph G

- $V(G) = \{A_i \mid i \in [n]\}$ events
- Edges situated such that

$$\Pr\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) \geq (1 - \epsilon) \Pr(A_i)$$

where S is any collection of non-neighbors of A_i and $\epsilon > 0$.

Compare with the negative dependence condition

$$\Pr\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) \leq \Pr(A_i).$$

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ϵ-Near Positive Dependency Graph for Checking Hats

Theorem (M 2016+)

The graph with vertex set $\{H_i \mid i \in [n]\}$ and no edges is an ϵ-near positive dependency graph with $\epsilon(n) = 1 - \frac{n-2}{n-1}$.

For hat-check events,
the ϵ-near positive dependence condition always holds.

ϵ-Near Positive Dependency Graph for Checking Hats

Theorem (M 2016+)

The graph with vertex set $\{H_i \mid i \in [n]\}$ and no edges is an ϵ-near positive dependency graph with $\epsilon(n) = 1 - \frac{n-2}{n-1}$.

For hat-check events,
the ϵ-near positive dependence condition always holds.

Once established, the positive dependence local lemma gives

$$\Pr\left(\bigcap_{i=1}^{n} \overline{H_i}\right) \leq \prod_{i=1}^{n} \left[1 - (1 - \epsilon(n)) \Pr(H_i)\right] = \left(1 - \frac{n-2}{n(n-1)}\right)^n \to \frac{1}{e}.$$

ϵ-Near Positive Dependency Graph for Checking Hats

Proof of edgeless ϵ-near positive dependency graph.
We need to establish

$$\Pr\left(H_n \,\middle|\, \bigcap_{j=1}^{k} \overline{H_j}\right) \geq \frac{n-2}{n-1} \Pr\left(H_n\right),$$

for all k,

ϵ-Near Positive Dependency Graph for Checking Hats

Proof of edgeless ϵ-near positive dependency graph.
We need to establish

$$\Pr\left(H_n \,\middle|\, \bigcap_{j=1}^{k} \overline{H_j}\right) \geq \frac{n-2}{n-1} \Pr\left(H_n\right),$$

for all k, which is equivalent to

$$\frac{n-2}{n-1}\left|\bigcap_{j=1}^{k} \overline{H_j}\right| \leq n \left|H_n \cap \bigcap_{j=1}^{k} \overline{H_j}\right|.$$

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ϵ-Near Positive Dependency Graph for Checking Hats

$$\frac{n-2}{n-1}\left|\bigcap_{j=1}^{k}\overline{H_j}\right| \leq n\left|H_n \cap \bigcap_{j=1}^{k}\overline{H_j}\right|$$

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ϵ-Near Positive Dependency Graph for Checking Hats

$$\frac{n-2}{n-1}\left|\bigcap_{j=1}^{k}\overline{H_j}\right| \leq n\left|H_n \cap \bigcap_{j=1}^{k}\overline{H_j}\right|$$

From the set of all arrangements in which

- none of the first k customers received their own hat,

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ϵ-Near Positive Dependency Graph for Checking Hats

$$\frac{n-2}{n-1}\left|\bigcap_{j=1}^{k}\overline{H_j}\right| \leq n\left|H_n \cap \bigcap_{j=1}^{k}\overline{H_j}\right|$$

From the set of all arrangements in which

- none of the first k customers received their own hat,

can we find a $\frac{n-2}{n-1}$ fraction of these

ϵ-Near Positive Dependency Graph for Checking Hats

$$\frac{n-2}{n-1}\left|\bigcap_{j=1}^{k}\overline{H_j}\right| \leq n\left|H_n \cap \bigcap_{j=1}^{k}\overline{H_j}\right|$$

From the set of all arrangements in which

- none of the first k customers received their own hat,

can we find a $\frac{n-2}{n-1}$ fraction of these

- that could have arisen from the switching operation in the previous proof?

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ϵ-Near Positive Dependency Graph for Checking Hats

Each numbered block contains $n-1$ arrangements, exactly one of which does not arise from the switching operation.
The leftover block contains no such arrangements and can be partitioned freely.

Customer	**1**	**2**	**3**	**4**
Block 1	4	3	2	1
	3	4	2	1
	2	3	4	1
Block 2	3	4	1	2
	4	3	1	2
	3	1	4	2
Block 3	2	1	4	3
	4	1	2	3
	2	4	1	3

Customer	**1**	**2**	**3**	**4**
Leftover	3	1	2	4
	2	3	1	4

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Summary

- Negative dependency gives a lower bound.
- ϵ-near positive dependency gives an upper bound.
- With luck, the bounds are asymptotically equal.

$$\prod_{i=1}^{n}(1 - x_i) \leq \Pr\left(\bigcap_{i=1}^{n} \overline{H_i}\right) \leq \prod_{i=1}^{n}\left[1 - (1 - \epsilon)\Pr(H_i)\right]$$

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$$\left(1 - \frac{1}{n}\right)^n \leq \Pr\left(\bigcap_{i=1}^{n} \overline{H_i}\right) \leq \left(1 - \frac{n-2}{n(n-1)}\right)^n$$

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Summary

- Negative dependency gives a lower bound.
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- With luck, the bounds are asymptotically equal.

$$\frac{1}{e} \leq \lim_{n\to\infty} \Pr\left(\bigcap_{i=1}^{n} \overline{H_i}\right) \leq \frac{1}{e}$$

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Thanks

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Further Information

- L. Lu and L. A. Székely, *Using the Lovász Local Lemma in the Space of Random Injections*, The Electronic Journal of Combinatorics, Volume 14(1) #63 (2007).
- L. Lu and L. A. Székely, *A New Asymptotic Enumeration Technique: The Lovász Local Lemma*, arXiv:0905.3983 [math.CO] (2011).
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