Beyond Infinity

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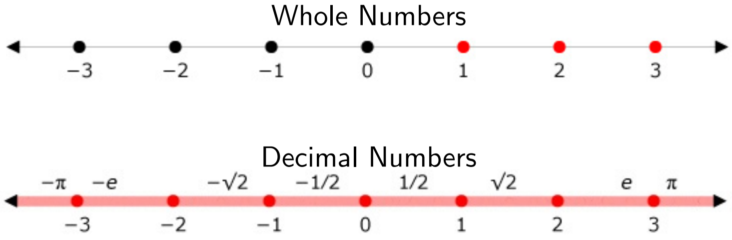
A Strange Thing

The collection of whole numbers and the collection of decimal numbers are both infinite, but are they really the same?



The whole numbers dot the landscape, while the decimal numbers fill it up.

Another Strange Thing

- What whole number comes after 1?
    - Easy, it's 2.
    - Next is 3.
    - Then 4.
    - And so on.
- The whole numbers can be *counted* or *listed* one after another.

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- What decimal number comes after 1?
  - Not 1.1, since 1.01 is closer.
  - 1.001 is even closer.
  - 1.0001 is closer still.
  - And so on.
- Maybe there are too many decimal numbers to list (whatever that means).

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Comparing Without Counting

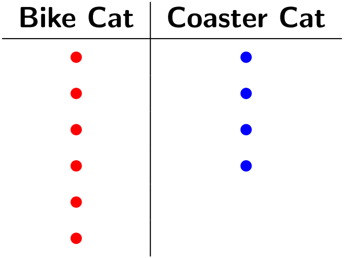
If we are going to compare the whole numbers with the decimal numbers, we'll need a strategy that doesn't involve counting.

Comparing Without Counting

- Both cats received lots of gifts.
- They need to know who received more.
- Cats don't know how to count.

Comparing Without Counting

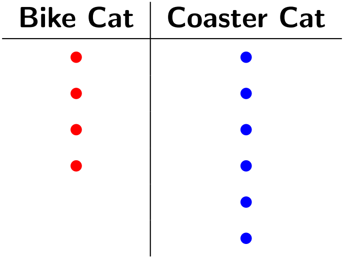
Strategy: Arrange the gifts in pairs with one from each cat.



- Unmatched gifts in Bike cat's column
- Bike cat got more gifts

Comparing Without Counting

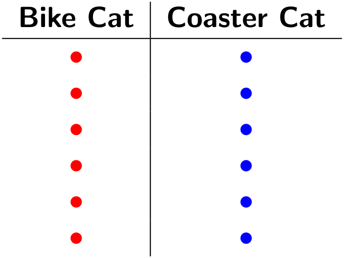
Strategy: Arrange the gifts in pairs with one from each cat.



- Unmatched gifts in Coaster cat's column
- Coaster cat got more gifts

Comparing Without Counting

Strategy: Arrange the gifts in pairs with one from each cat.



- No unmatched gifts in either column
- Each cat received the same number of gifts
- We'll say the collections of gifts are *matched*.

Comparing Without Counting

Morals:

1. If two collections can be matched, then they are the same size.
2. If two collections cannot be matched, then one of them is larger.

Comparing Without Counting

What does this have to do with our problem?

- If we can match the whole numbers with the decimal numbers, then they must be the same size.
- Whenever we can match a collection with the whole numbers, we say that collection is *countable*.

Comparing Without Counting

What does this have to do with our problem?

- If they cannot be matched, then the collection of decimal numbers must be larger.
- Whenever a collection is too big to be matched with the whole numbers, we say that collection is *uncountable*.

Countable Collections: Even Whole Numbers

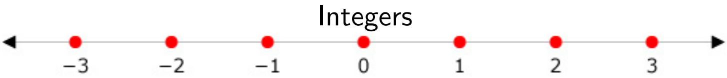
Whole	**Even**
1	
2	
3	
$\vdots$	
100	
101	
102	
$\vdots$	

Countable Collections: Even Whole Numbers

Whole	**Even**
1	2
2	4
3	6
⋮	⋮
100	200
101	202
102	204
⋮	⋮

The even whole numbers can be matched with the whole numbers, so they are the same size.

"Half of countable is still countable."



- The integers are "twice as big" as the whole numbers, extending infinitely both forward and backward.

Countable Collections: Integers

Whole	Integer
1	0
2	1
3	-1
4	2
5	-2
⋮	⋮
100	50
101	-50
102	51
103	-51
⋮	⋮

The integers can be matched with the whole numbers, so they are the same size.

"Twice countable is still countable."

Countable Collections: Fractions

We can play the same game with the collection of fractions as we did with the collection of decimals.

- What fraction comes after 0?
  - Not $\frac{1}{2}$, since $\frac{1}{4}$ is closer.
  - $\frac{1}{8}$ is even closer.
  - $\frac{1}{16}$ is closer still.
  - And so on.
- Between *any* two fractions, there are infinitely-many fractions.
- There are decimal numbers that aren't fractions, however.
  - $\sqrt{2}$, π, e, $\ln(3)$, and many others

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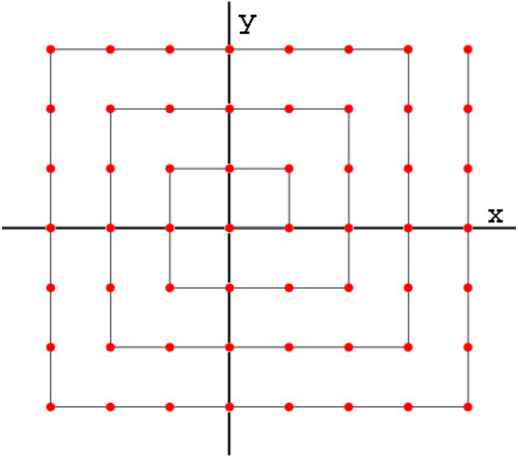
Countable Collections: Fractions

The size of the collection of fractions is "countable times countable".

- A fraction is an integer divided by an integer (e.g. $\frac{-5}{3}$).
- There are countably-many choices for the numerator and countably-many choices for the numerator.

Countable Collections: Fractions

We can order the fractions in a clever way.

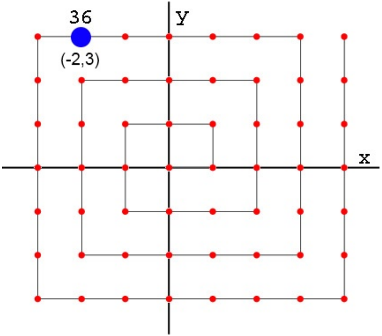


- Let x be the numerator and y be the denominator.
- Spiral out to visit all the fractions.

Countable Collections: Fractions

For example:

- Dot number 36 is at coordinate $(-2, 3)$.
- Match the whole number 36 with the fraction $\frac{-2}{3}$.



The fractions can be matched with the whole numbers, so they are the same size.

"Countable times countable is still countable."

An Uncountable Collection: Decimals

Finally, let's show that the collection of decimals really is uncountable.

- We're going to show that the collection of decimals between 0 and 1 (excluding 1) is already uncountable.
- Start by assuming we've managed to match these decimals with the whole numbers (i.e. that they are countable).
- We'll discover there's something horribly wrong with our list.
- *Any* attempt to list these decimals will be doomed to failure.
- We'll be forced to conclude that the decimals are uncountable.

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An Uncountable Collection: Decimals

A typical list might look like this.

Whole	**Decimal**
1	.394820...
2	.056733...
3	.870356...
4	.356734...
5	.745695...
6	.153455...
⋮	⋮

An Uncountable Collection: Decimals

Highlight all the digits on the diagonal.

Whole	Number
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2	.056733...
3	.870356...
4	.356734...
5	.745695...
6	.153455...
⋮	⋮

Call this decimal number the "diagonal decimal".

.350795...

An Uncountable Collection: Decimals

Form the "death knell decimal" from the diagonal decimal by adding 1 to each digit (wrapping 9's back to 0's).

$$\begin{aligned} \text{Diagonal Decimal:} &\quad .350795\ldots \\ \text{Death Knell Decimal:} &\quad .461806\ldots \end{aligned}$$

Where does the death knell decimal appear on our list?

An Uncountable Collection: Decimals

Diagonal Decimal:	.350795 . . .
Death Knell Decimal:	.461806 . . .

- Is it in row 1?
  - No. The **first** digit of row **1**'s entry is a 3.
- Is it in row 2?
  - No. The **second** digit of row **2**'s entry is a 5.
- Is it in row 3?
  - No. The **third** digit of row **3**'s entry is a 0.
- And so on.

No matter which row you check, the death knell decimal differs in at least one place.

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Where does the death knell decimal appear on our list?

It doesn't.

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It doesn't.

An Uncountable Collection: Decimals

What does this mean?

- Our list, which supposedly listed all decimal numbers (between 0 and 1), is missing a number.
- The list can never be "patched up" by adding missing decimals, because we can always perform the diagonal operation again.
- Since no list will ever contain all the decimals, there must be more decimals than whole numbers.

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The decimals constitute a larger kind of infinity than the whole numbers.

How Much Bigger?

How much bigger is uncountable than countable?

- Finite : Countable :: Countable : Uncountable

How Much Bigger?

Finite : Countable

- Take a countable set (like the whole numbers).
- Remove the first 100 numbers.
- You still have countably infinitely-many left.

Removing finitely-many things from a countably infinite set does not make a dent.

How Much Bigger?

Countable : Uncountable

- Take an uncountable set (like the decimal numbers).
- Remove countably-many numbers.
- You still have uncountably-many left.

Removing countably infinitely-many things from an uncountable set does not make a dent.

How Much Bigger?

In particular

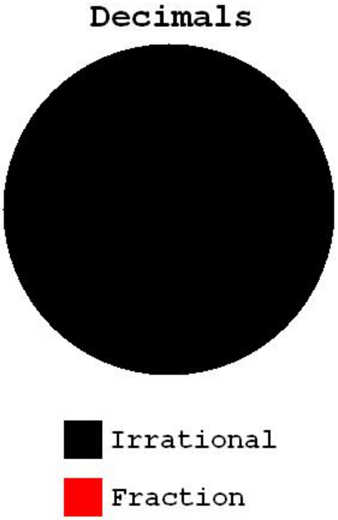
- The fractions are countable.
- The decimals are uncountable.
- So, the decimals with all the fractions removed is still an uncountable collection.

What we have remaining are the *irrational* numbers.

- $\sqrt{2}$, π, e, $\ln(3)$, and many others

How Much Bigger?

Essentially all the decimal numbers are irrational.

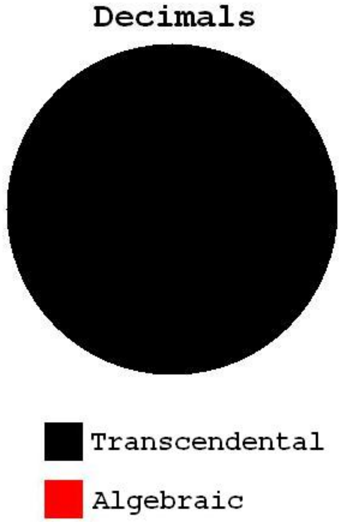


How Much Bigger?

- The collection of roots of numbers (square roots, cube roots, etc.) is countable.
- We can remove these as well, and we're left with the *transcendental numbers*.
  - π, e, $\ln(2)$, and many others.

How Much Bigger?

Essentially all the decimal numbers are transcendental.

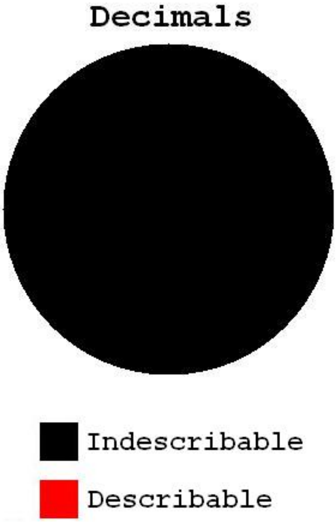


How Much Bigger?

- The *describable numbers* are all the numbers that can be described using finitely-many characters of any kind (linguistic, mathematical, etc.).
  - -1, $\frac{1}{2}$, $\sqrt{2}$, $\sum_{n=0}^{\infty} \frac{1}{n!}$, "the ratio of circumference to diameter"
- The collection of describable numbers is countable.
- We can remove these from the decimal numbers, and we're left with the *indescribable numbers*.

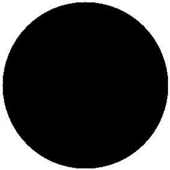
How Much Bigger?

Essentially all the decimal numbers are indescribable.



If all humankind pooled all its resources for all time,

they could not describe even a single indescribable number.



Constructing Even Larger Infinities

- So far, we've seen two kinds of infinities (countable and uncountable).
- There is a machine (called the *power set*) that takes one infinity and produces a larger one.
- Using the machine over and over gives you an endless stream of larger infinities.

Let's see how the power set works on a finite collection.

Constructing Even Larger Infinities

The *power set* takes a collection and gives you all possible combinations of the members (order doesn't matter).

Members of Original Set:	$1, 2, 3$
Members of Power Set:	$\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

- The original set contains numbers, while the power set contains *collections* of numbers.
- The original set has 3 numbers, while the power set contains 8 collections.

Constructing Even Larger Infinities

What about the power set of a countably infinite set like the whole numbers?

$$\begin{aligned}
\text{Members of Original Set:}\quad & 1, 2, 3, \ldots \\
\text{Members of Power Set:}\quad & \{\}, \\
& \{1\}, \{2\}, \{3\}, \ldots, \\
& \{1, 2\}, \{1, 3\}, \{1, 4\}, \ldots, \\
& \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\} \ldots, \\
& \vdots
\end{aligned}$$

Each row (except the first) has countably infinitely-many things in it, and there are countably infinitely-many rows.

Constructing Even Larger Infinities

Why might we expect the power set to be larger than the original set?

Whole	**Power Set**
1	$\{1\}$
2	$\{2\}$
3	$\{3\}$
4	$\{4\}$
5	$\{5\}$
$\vdots$	$\vdots$
100	$\{100\}$
101	$\{101\}$
$\vdots$	$\vdots$

We've matched the whole numbers with a tiny, tiny portion of the power set.

Constructing Even Larger Infinities

Here is an endless stream of progressively larger infinite sets.

$$\mathbb{W}, \mathcal{P}(\mathbb{W}), \mathcal{P}(\mathcal{P}(\mathbb{W})), \ldots$$

Each new infinity is unapproachable by smaller infinities.

$$\text{Finite} : \text{Countable} :: \mathcal{P}^n(\mathbb{W}) : \mathcal{P}^{n+1}(\mathbb{W})$$

Conclusion

1. Infinity is only the beginning.
2. The numbers we can describe are nothing compared to the numbers that exist.
3. Mathematics is a creative endeavor.

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Thanks

Contact Me
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