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July, 1994
file: "/bell/bellMoser.tex"
update(s): August, 1995

The Moser-Wyman expansion of the Bell numbers. Let B_n be the n-th Bell number, and r be the solution of the equation

$$re^r = n.$$

Then uniformly for $h = O(\log n)$

$$\begin{aligned} B_{n+h} = {} & \frac{(n+h)!}{r^{n+h}} \; \frac{\exp(e^r - 1)}{(2\pi B)^{1/2}} \; \times \\ & \times \; \Big(1 + \frac{P_0 + hP_1 + h^2P_2}{e^r} + \frac{Q_0 + hQ_1 + h^2Q_2 + h^3Q_3 + h^4Q_4}{e^{2r}} + O(e^{-3r})\Big), \end{aligned}$$

as $n \to \infty$, where

$$B = (r^2 + r)e^r$$

$$P_0 = -\frac{2r^4 + 9r^3 + 16r^2 + 6r + 2}{24r(r+1)^3}$$

$$P_1 = -\frac{r^2 + 3r + 1}{2r(r+1)^2}$$

$$P_2 = -\frac{1}{2r(r+1)}$$

$$Q_0 = \frac{4 + 24r + 100r^2 - 636r^3 - 588r^4 - 384r^5 - 143r^6 - 12r^7 + 4r^8}{1152r^2(r+1)^6}$$

$$Q_1 = \frac{6 + 32r + 56r^2 + 135r^3 + 101r^4 + 37r^5 + 6r^6}{48r^2(r+1)^5}$$

$$Q_2 = \frac{20 + 90r + 190r^2 + 105r^3 + 20r^4}{48r^2(r+1)^4}$$

$$Q_3 = \frac{5 + 15r + 5r^2}{12r^2(r+1)^3}$$

$$Q_4 = \frac{1}{8r^2(r+1)^2}$$

The original reference for asymptotic expansions of B_n is Moser and Wyman, *Transactions of the Royal Society of Canada* 49 (1955) 49-54.

Sketch of the proof. Start with Cauchy's formula:

$$\frac{B_{n+h}}{(n+h)!} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{\exp(e^z - 1)dz}{z^{n+h+1}}.$$

Let $\delta = Ce^{-r/2}$, C a constant at our disposal. Substitute $z = re^{i\theta}$ and split the interval of integration to obtain

$$\begin{aligned}\frac{B_{n+h}r^{n+h}}{(n+h)!} &= \frac{1}{2\pi} \int_0^{2\pi} \exp\{e^{re^{i\theta}} - 1 - (n+h)i\theta\}d\theta \\ &= \int_{-\delta}^{+\delta} \cdots + \int_{\delta}^{2\pi-\delta} \cdots \\ &= I_1 + I_2.\end{aligned}$$

We now evaluate I_1 and I_2 separately. First we prove the second integral negligible. Using

$$\begin{gathered}\cos\delta \le 1 - \frac{1}{3}\delta^2, \ \ 0 \le \delta \le 3^{1/2} \\ e^x \le 1 + \frac{1}{2}x, \ \ -1 \le x \le 0 \\ \operatorname{Re} e^z = e^{\operatorname{Re} z}\cos(\operatorname{Im} z)\end{gathered}$$

we have

$$\begin{aligned}\operatorname{Re} e^{r(e^{i\theta}-1)} &= e^{r(\cos\theta - 1)}\cos(r\sin\theta) \\ &\le e^{r(\cos\theta-1)} \\ &\le e^{r(cos\delta - 1)} \ \text{ if } \ \delta \le \theta \le 2\pi - \delta \\ &\le 1 + \frac{1}{2}r(\cos\delta - 1) \\ &\le 1 - \frac{1}{6}r\delta^2,\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re} e^{re^{i\theta}} - e^r &= e^r(\operatorname{Re} e^{r(e^{i\theta}-1} - 1) \\ &\le -\frac{1}{6}r\delta^2 e^r \\ &= -\frac{1}{6}C^2 r,\end{aligned}$$

and, since $|\exp(z)| = \exp(\operatorname{Re} z)$,

$$\begin{aligned}\big|I_2\big| &\le \exp(e^r - 1) \operatorname*{Max}_{\delta\le\theta\le 2\pi-\delta} \exp(\operatorname{Re} e^{re^{i\theta}} - e^r) \\ &\le \exp(e^r - 1 - \frac{1}{6}C^2 r). \\ &= \frac{\exp(e^r-1)}{(2\pi B)^{1/2}} O(e^{-3r}),\end{aligned}$$

provided $1/2 - C^2/6 < -3$.

Recall from the theory of polynomials of binomial type

$$\exp\{x(e^u - 1)\} = \sum_{m=0}^{\infty} \frac{u^m}{m!} \sum_k S(m,k)x^k$$
$$= \sum_{m=0}^{\infty} \frac{u^m}{m!} S_m(x),$$

the Stirling polynomials, whence

$$e^{re^{i\theta}} = e^r \sum_{m=0}^{\infty} \frac{(i\theta)^m}{m!} S_m(r).$$

Let $B = e^r S_2(r)$, and let r be determined by the condition $e^r S_1(r) = n$. Putting all this together, we find

$$I_1 = \frac{\exp(e^r - 1)}{2\pi} \int_{-\delta}^{+\delta} \exp\{e^{re^{i\theta}} - e^r - (n+h)i\theta\} d\theta$$
$$= \frac{\exp(e^r - 1)}{2\pi} \int_{-\delta}^{+\delta} \exp\{-hi\theta - \frac{1}{2}B\theta^2 + W\} d\theta,$$

say, and we can apply Taylor's theorem with remainder as given for example on page 177 of Ahlfors to the function $g(z) = \exp(re^z)$ to write W thus

$$W = e^r \sum_{m=3}^{7} \frac{(i\theta)^m}{m!} S_m(r) + \frac{\theta^8}{2\pi i} \oint_{|z|=R} \frac{\exp\{re^z\}dz}{z^8(z - i\theta)}, \quad |\theta| \le \delta,$$

provided $R > \delta$. The final term on the right of the last equation is $O(r^8 e^r \theta^8)$, as can be seen by choosing $R = 1/r$. The strategy for I_1 is simply to expand $\exp\{-hi\theta - \frac{1}{2}B\theta^2 + W\}$ into a finite sum of terms of the form $ce^{mr}h^a\theta^b e^{-\frac{1}{2}B\theta^2}$, c rational in r and m, a, b integers; and of the form $O(r^d e^{mr}\theta^b e^{-\frac{1}{2}B\theta^2})$. For each fixed b we have

$$\int_{|\theta| \ge \delta B^{1/2}} \theta^b e^{-\frac{1}{2}\theta^2} d\theta = O(1)(\delta B^{1/2})^b e^{-\frac{1}{2}B\delta^2}$$
$$= O(r^b e^{-\frac{1}{2}C^2 r^2}),$$

the big-oh depending on both b and C. Thus each such term can be explicitly integrated:

$$\int_{-\delta}^{+\delta} \theta^{2b} e^{-\frac{1}{2}B\theta^2} d\theta = B^{-1/2} \int_{-\delta B^{1/2}}^{+\delta B^{1/2}} \frac{\theta^{2b}}{B^b} e^{-\frac{1}{2}\theta^2} d\theta$$
$$= \int_{-\infty}^{+\infty} \cdots - \int_{|\theta| \ge \delta B^{1/2}}$$
$$= \sqrt{\frac{2\pi}{B}} \Big(\frac{(2b-1)(2b-3)\cdots(3)(1)}{B^b} + O(e^{-3r}) \Big).$$

To obtain the accuracy given in the theorem, it is necessary to use the following: (odd θ denotes a sum of odd powers of θ times rational functions of r)

$$\exp(W) = 1 + W + \frac{1}{2}W^2 + \frac{1}{6}W^3 + \frac{1}{24}W^4 + O(W^5)$$

$$W = \cdots \text{ as given earlier } \cdots$$

$$W^2 = e^{2r}\Big(\frac{(i\theta)^6}{(3!)^2}S_3(r)^2 + 2\frac{(i\theta)^8}{3!5!}S_3(r)S_5(r) + \frac{(i\theta)^8}{(4!)^2}S_4(r)^2 + \text{odd}\,\theta + O(r^{10}\theta^{10})\Big)$$

$$W^3 = e^{3r}\Big(3\frac{(i\theta)^{10}}{(3!)^2 4!}S_3(r)^2 S_4(r) + \text{odd}\,\theta + O(r^{12}\theta^{12})\Big)$$

$$W^4 = e^{4r}\Big(\frac{(i\theta)^{12}}{(3!)^4}S_3(r)^4 + \text{odd}\,\theta + O(r^{14}\theta^{14})\Big)$$

$$W^5 = e^{5r}\Big(\frac{(i\theta)^{15}}{(3!)^5}S_3(r)^5 + O(r^{16}\theta^{16})\Big)$$

$$\exp(-hi\theta) = 1 - hi\theta + \frac{1}{2}h^2(i\theta)^2 - \frac{1}{6}h^3(i\theta)^3 + \frac{1}{24}h^4(i\theta)^4 - \frac{1}{120}h^5(i\theta)^5 + O(r^6\theta^6).$$

Note that odd powers of θ contribute nothing to the final results. Attached to the permanently filed hardcopy of this writeup are three pages of handwritten notes and a short Maple program which show the computation of the P_i and Q_i. Similar expansions out to e^{-Kr}, for integer K even larger than 3, are possible by this same method, although very complicated to compute explicitly and of no practical use encountered thus far.

An alternate formulation. There is a second asymptotic expansion for B_{n+h} which may be useful. We eliminate the quantity $(n+h)!$ by using Stirling's formula:

$$(n+h)! = n^{n+h}e^{-n}\sqrt{2\pi}\Big(1 + \frac{6h^2+6h+1}{n} + \frac{36h^4+24h^3-24h^2-12h+1}{288n^2} + O(h^6/n^3)\Big).$$

The resulting expansion is:

$$B_{n+h} = \frac{e^{rh}}{(r+1)^{1/2}}\exp\{n(r-1+1/r)-1\} \times \times \Big(1 + \frac{\hat{P}_0 + h\hat{P}_1 + h^2\hat{P}_2}{e^r} + \frac{\hat{Q}_0 + h\hat{Q}_1 + h^2\hat{Q}_2 + h^3\hat{Q}_3 + h^4\hat{Q}_4}{e^{2r}} + O(r^3e^{-3r})\Big), \qquad (*)$$

in which

$$\hat{P}_0 = -\frac{10r + 7r^2 + 2r^3}{24(r+1)^3}$$

$$\hat{P}_1 = -\frac{1}{2(r+1)^2}$$

$$\hat{P}_2 = \frac{1}{2(r+1)}$$

$$\hat{Q}_0 = -\frac{864r + 860r^2 + 556r^3 + 199r^4 + 20r^5 - 4r^6}{1152(r+1)^6}$$

$$\hat{Q}_1 = \frac{-20 + 42r + 43r^2 + 20r^3 + 4r^4}{48(r+1)^5}$$

$$\hat{Q}_2 = \frac{42 + 2r - 7r^2 - 2r^3}{48(r+1)^4}$$

$$\hat{Q}_3 = -\frac{7 + 2r}{12(r+1)^3}$$

$$\hat{Q}_4 = \frac{1}{8(r+1)^2}$$

Another way to derive an asymptotic expansion for the Bell numbers is that used by deBruijn, and earlier by Epstein although the latter made an error. In this approach we start with Dobinski's sum

$$B_{n+h} = \frac{1}{e}\sum_{j=0}^{\infty} \frac{j^{n+h}}{j!},$$

and follow the usual method for estimating a sum: find the maximum term and estimate the terms near it. In the above the maximum takes place for j near m, where $m \log m = n$. After substituting $j = m + a$ it is necessary to estimate a sum over a containing $e^{-ca^2}a^b$. One typically does the latter estimation with an integral, but using various formulas from the theory of theta series (see Rademacher) will allow one greater control on the error bounds. When this approach is followed, the expansion which emerges is of the form (*) above. This provides a way of checking the calculations. Notice that the error term in (*) is $O(r^3 e^{-3r})$ rather than the earlier $O(e^{-3r})$.

The variance of the partition lattice Π_n. We now give an asymptotic formula for the variance of the number of blocks in a partition of an n-element set. Using the standard recursion for the Stirling numbers $S(n,k)$ to express $\sum k^2 S(n,k)$ in terms of the Bell numbers, as found in Harper's paper, *Ann. Math. Stat.* 38 (1967) 171-215, we have

$$\sigma^2 = \frac{B_{n+2}}{B_n} - \big(\frac{B_{n+1}}{B_n}\big)^2 - 1.$$

Using the above expansion for B_n we have computed that

$$\frac{B_{n+2}}{B_n} - \big(\frac{B_{n+1}}{B_n}\big)^2 = \frac{n}{r(r+1)} + \frac{r(r-1)}{2(r+1)^4} + O(e^{-r}),$$

using the Maple program stored in the file "/bell/var." This formula amplifies the one given on page 412 by Harper and attributed by him to J. Haigh, in which the latter two terms on the right appear as $o(1)$. It is interesting that we have a "closed formula" for σ^2 which is accurate to within $o(1)$.

The asymptotic solution of Engel's conjecture.

Engel has conjectured that the average number of blocks in a partition of an n-set is a concave function of n. In terms of the Bell numbers this conjecture asserts

$$\frac{B_{n+1}}{B_n} \geq \frac{1}{2}\big(\frac{B_{n+2}}{B_{n+1}} + \frac{B_n}{B_{n-1}}\big).$$

Using our asymptotic formula for B_{n+h} we have computed

$$\begin{aligned}
&\frac{B_{n+1}}{B_n} - \frac{1}{2}\big(\frac{B_{n+2}}{B_{n+1}} + \frac{B_n}{B_{n-1}}\big) \\
&\qquad = \frac{-3Q_3 - 6Q_4 + P_2^2 + 3P_1P_2 - 2P_2/r}{e^r} + O(e^{-2r}) \\
&\qquad = \frac{1/2}{(r+1)^3 e^r} + O(e^{-2r}),
\end{aligned}$$

thus verifying Engel's conjecture for all sufficiently large n. This will appear (galley proofs received in July, 1995) in the "NOTE: Engel's inequality for Bell numbers," *JCT*, Series A.

The asymptotic solution of Canfield's conjecture.

Canfield has conjectured that the average number of singleton blocks in a partition of an n-set is an increasing function of n. This conjecture, and a considerable generalization of it, are proven in "Log concavity and a related property of the cycle index polynomials," by Bender and Canfield; the latter has been accepted for publication in *JCT*, Series A.

In terms of the Bell numbers this conjecture asserts that the sequence nB_{n-1}/B_n increases, and in fact we have

$$\begin{aligned}
\frac{(n+1)B_n}{B_{n+1}} - \frac{nB_{n-1}}{B_n} &= r\Big(\frac{-2P_1}{e^r} + O(e^{-2r})\Big) \\
&= \frac{1+O(r^{-1})}{e^r},
\end{aligned}$$

thus proving the conjecture for all sufficiently large n.