Connected Balanced Subgraphs in Random Regular Multigraphs Under the Configuration Model*

Linyuan Lu Austin Mohr
László Székely
Department of Mathematics
University of South Carolina
1523 Greene Street, Columbia, SC 29208
lu@mailbox.sc.edu, mohrat@email.sc.edu, laszlo@mailbox.sc.edu

Abstract

Our previous paper [9] applied a lopsided version of the Lovász Local Lemma that allows negative dependency graphs [5] to the space of random matchings in K_{2n}, deriving new proofs to a number of results on the enumeration of regular graphs with excluded cycles through the configuration model [3]. Here we extend this from excluded cycles to some excluded balanced subgraphs, and derive asymptotic results on the probability that a random regular multigraph from the configuration model contains at least one from a family of balanced subgraphs in question.

1 The Tool

In [9] we proved the following theorem on extensions of (partial) matchings that allows (among other things) proving asymptotic enumeration results about regular graphs through the configuration model.

Theorem 1 *Let Ω be the uniform probability space of perfect matchings in the complete graph K_N (N even) or the complete bipartite graph $K_{N,N'}$ (with $N \leq N'$). Let $r = r(N)$ be a positive integer and $1/16 > \epsilon = \epsilon(N) > 0$*

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as N approaches infinity. Let $\mathcal{M} = \mathcal{M}(N)$ be a collection of (partial) matchings in K_N or $K_{N,N'}$, respectively, such that none of these matchings is a subset of another. For any $M \in \mathcal{M}$, let A_M be the event consisting of perfect matchings extending M. Set $\mu = \mu(N) = \sum_{M\in\mathcal{M}} \Pr(A_M)$. Suppose that $\mathcal{M}$ satisfies

1. $|M| \leq r$, *for each* $M \in \mathcal{M}$.
2. $\Pr(A_M) < \epsilon$ *for each* $M \in \mathcal{M}$.
3. $\sum_{M':A_{M'}\cap A_M=\emptyset} \Pr(A_{M'}) < \epsilon$ *for each* $M \in \mathcal{M}$.
4. $\sum_{M:uv\in M\in\mathcal{M}} \Pr(A_M) < \epsilon$ *for each single edge* uv.
5. $\sum_{H\in\mathcal{M}_F} \Pr_{N-2r}(A_H) < \epsilon$ *for each* $F \in \mathcal{M}$.

Then, if $r\epsilon = o(1)$, *we have*

$$\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M}) = e^{-\mu+O(r\epsilon\mu)}, \tag{1}$$

and furthermore, if $r\epsilon\mu = o(1)$, *then*

$$\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M}) = \Big(1 + O(r\epsilon\mu)\Big)e^{-\mu}. \tag{2}$$

In the theorem above $\Pr(A_M)$ denotes the probability according to the counting measure, and $\Pr_{N-2r}(A_H)$ indicates the probability of A_H in a setting, when $2r$ of the N vertices (none of them is an endpoint of an edge in the partial matching H) are eliminated, and the probability is considered in this smaller instance of the problem.

2 The Configuration Model and the Enumeration of d-Regular Graphs

For a given sequence of positive integers with an even sum, $(d_1, d_2, \ldots, d_n) = \mathbf{d}$, the *configuration model of random multigraphs with degree sequence* $\mathbf{d}$ is defined as follows [3].

1. Let us be given a set U that contains $N = \sum_{i=1}^n d_i$ distinct mini-vertices. Let U be partitioned into n classes such that the ith class consists of d_i mini-vertices. This ith class will be associated with vertex v_i after identifying its elements through a *projection*.
2. Choose a random matching M of the mini-vertices in U uniformly.

3. Define a random multigraph G associated with M as follows: For any two (not necessarily distinct) vertices v_i and v_j, the number of edges joining v_i and v_j in G is equal to the total number of edges in M between mini-vertices associated with v_i and mini-vertices associated with v_j.

The configuration model of random d-regular multigraphs on n vertices is the instance $d_1 = d_2 = \cdots = d_n$, where nd is even.

Bender and Canfield [2], and independently Wormald, showed in 1978 that for any fixed d, with nd even, the number of d-regular graphs is

$$(\sqrt{2} + o(1))e^{\frac{1-d^2}{4}} \left(\frac{d^d n^d}{e^d (d!)^2} \right)^{\frac{n}{2}}. \tag{3}$$

Bollobás [3] introduced probability to this enumeration problem by defining the configuration model, and brought the result (3) to the alternative form

$$(1 + o(1))e^{\frac{1-d^2}{4}} \frac{(dn-1)!!}{(d!)^n}, \tag{4}$$

where the term $(1 + o(1))e^{\frac{1-d^2}{4}}$ in (4) can be explained as the probability of obtaining a simple graph after the projection. (The semifactorial $(dn-1)!! = \frac{(dn)!}{(dn/2)!2^{dn/2}}$ equals the number of perfect matchings on dn elements, and $(d!)^n$ is just the number of ways matchings can yield the same simple graph after projection. Non-simple graphs, unlike simple graphs, can arise with different frequencies.) Bollobás also extended the range of the asymptotic formula to $d < \sqrt{2 \log n}$, which was further extended to $d = o(n^{1/3})$ by McKay [10] in 1985. The strongest result is due to McKay and Wormald [11] in 1991, who refined the probability of obtaining a simple graph after the projection to $(1 + o(1))e^{\frac{1-d^2}{4} - \frac{d^3}{12n} + O(\frac{d^2}{n})}$ and extended the range of the asymptotic formula to $d = o(n^{1/2})$. Wormald's Theorem 2.12 in [15] (originally published in [14]) asserts that for any fixed numbers $d \geq 3$ and $g \geq 3$, the number of labelled d-regular graphs with girth at least g, is

$$(1 + o(1))e^{-\sum_{i=1}^{g-1} \frac{(d-1)^i}{2i}} \frac{(dn-1)!!}{(d!)^n}. \tag{5}$$

[9] reproved the following theorem of McKay, Wormald and Wysocka [12] using Theorem 1, under a slightly stronger condition than $(d-1)^{2g-3} = o(n)$ in [12]: (note that a power of g in (6) only restricts a second term in the asymptotic series of the bound on g):

Theorem 2 *In the configuration model, assume $d \geq 3$ and*

$$g^6(d-1)^{2g-3} = o(n). \tag{6}$$

Then the probability that the random d-regular multigraph has girth at least $g \geq 1$ is $(1+o(1))\exp\Big(-\sum_{i=1}^{g-1} \frac{(d-1)^i}{2i}\Big)$, and hence the number of d-regular graphs on n vertices with girth at least $g \geq 3$ is

$$(1+o(1))e^{-\sum_{i=1}^{g-1} \frac{(d-1)^i}{2i}} \frac{(dn-1)!!}{(d!)^n}.$$

(The case $g = 3$ means that the random d-regular multigraph is actually a simple graph.) Furthermore, the number of d-regular graphs not containing cycles whose length is in a set $\mathcal{C} \subseteq \{3, 4, ..., g-1\}$, is

$$(1+o(1))e^{-\frac{d-1}{2}-\frac{(d-1)^2}{4}-\sum_{i\in\mathcal{C}} \frac{(d-1)^i}{2i}} \frac{(dn-1)!!}{(d!)^n}.$$

This is a special case of a more general result. The following definitions are used in random graph theory [1]. The *excess* of a graph G, denoted by $\kappa(G)$, is $|E(G)| - |V(G)|$. A graph G is *balanced*, if $\kappa(H) < \kappa(G)$ for any proper subgraph H with at least one vertex. We first prove the following Lemma.

Lemma 3 *Suppose that G is a connected balanced simple graph with $\kappa(G) \geq 0$. Then the number of subgraphs H with $\kappa(H) = \kappa(G) - 1$ is at most $2|V(G)|^2$.*

Proof: First we claim that G has no leaf vertex. Otherwise, if v is a leaf vertex, then $\kappa(G - v) = \kappa(G)$, a contradiction.

Let H be a subgraph of G with $\kappa(H) = \kappa(G) - 1$. If $V(H) = V(G)$, then H is obtained by deleting one edge from G. The number of such H's is $|E(G)|$. Now we assume $V(H) \neq V(G)$. For any vertex set S, let $\Gamma(S)$ be the neighborhood of S in G. We define a sequence of graphs $H_0, H_1, H_2, \ldots$ as follows. Let $H_0 = H$. For $i \geq 1$, if $V(H_{i-1}) \neq V(G)$, we define the graph H_i as follows: $V(H_i) = V(H_{i-1}) \cup \Gamma(V(H_{i-1}))$ and $E(H_i) = E(H_{i-1}) \cup \{uv\colon u \in V(H_{i-1}), v \in \Gamma(V(H_{i-1})), \text{ and } uv \in E(G)\}$. Let H_r be the last graph in the sequence. We have $V(H_r) = V(G)$. Observe

$$\kappa(H) = \kappa(H_0) \leq \kappa(H_1) \leq \kappa(H_2) \cdots \leq \kappa(H_r) \leq \kappa(G).$$

Since $\kappa(H) = \kappa(G) - 1$, equalities hold for all but at most one in the chain above. We have $|\Gamma(V(H_i)) \setminus V(H_i)| \leq 2$ for all $i \leq r-1$, as G has no leaf.

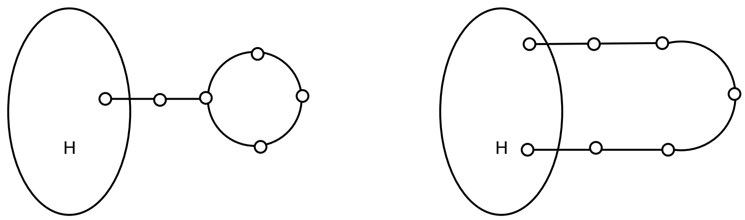

Figure 1: $G - H$ is either a ρ-shape or a path when $\kappa(H) = \kappa(G) - 1$.

It is easy to check that the difference of G and H either forms a ρ-shape or is a path as shown in Figure 1. An H with a ρ-shape may occur at most $2|E(G)|$ times, an H with a path may occur at most most $\binom{|V(G)|}{2}$ times.

Finally, $|E(G)| + 2|E(G)| + \binom{|V(G)|}{2} \leq 2|V(G)|^2$. $\square$

Let $\mathcal{G}$ be a family of connected balanced simple graphs with excess κ. We would like to estimate the probability that a random d-regular multigraph contains no graph in $\mathcal{G}$. Given a simple graph G, let $|\mathrm{Aut}(G)|$ be the number of automorphisms of G. For any $k \geq 2$, let $a_k(G)$ be the number of vertices with degree at least k. We define a polynomial $f_G(d) = \prod_{k=2}^{\infty}(d-k+1)^{a_k(G)} = \prod_{v\in V(G)} \binom{d-1}{d_v-1}(d_v-1)! = \frac{1}{d^{|V(G)|}} \prod_{v\in V(G)} \binom{d}{d_v} d_v! \leq (d-1)^{2|E(G)|-|V(G)|}$. We have the following theorem.

Theorem 4 *Let $\mathcal{G}$ be a family of connected balanced simple graphs with non-negative excess κ. Set $r = \max_{G\in\mathcal{G}} |E(G)|$. In the configuration model, assume $d \geq 3$ and*

$$\frac{r^3}{n} \sum_{G\in\mathcal{G}} (d-1)^{|E(G)|-1} = o(1) \text{ and } \ell = \frac{r^3 d}{n^{\kappa+1}} \Big(\sum_{G\in\mathcal{G}} (d-1)^{|E(G)|-1}\Big)^2 = o(1). \tag{7}$$

Then the probability that the random d-regular multigraph arising from the configuration model contains no subgraph in $\mathcal{G}$ is

$$(1 + O(\ell)) \exp\Big(-\sum_{G\in\mathcal{G}} \frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}\Big).$$

Proof: Let $\epsilon = \frac{Kr^2}{n} \sum_{G\in\mathcal{G}} (d-1)^{|E(G)|-1}$ with a large constant K. The first condition makes sure $r\epsilon = o(1)$, the second condition makes sure $r\epsilon\mu = O(\ell) = o(1)$ in Theorem 1.

For any $G \in \mathcal{G}$, let M_G be the family of (partial) matchings of U whose projection is a copy of G. Suppose that G has s vertices $v_1, \ldots, v_s$ and t edges $e_1, \ldots, e_t$. For $1 \leq i \leq s$, let C_i be the class of d mini-vertices associated to v_i and Q_i be an (ordered) queue of d_{v_i} mini-vertices in C_i. Let $\mathcal{C}$ be the parameter space of all possible $(C_1, \ldots, C_s, Q_1, \ldots, Q_s)$. We define a mapping $\psi\colon \mathcal{C} \to M_G$ as follows. For $1 \le j \le t$, suppose that the edge e_j has two end-vertices v_{j_1} and v_{j_2}. We pop a mini-vertex x_j from the queue Q_{j_1}, pop a mini-vertex y_j from the queue Q_{j_2}, and join $x_j y_j$. Denote by M the collection of edges $\{x_j y_j\}_{1 \le j \le t}$. Clearly M forms a partial matching whose projection is a copy of G. We define $\psi(C_1, \ldots, C_s, Q_1, \ldots, Q_s) = M$. Since every partial matching in M_G can be constructed in this way, ψ is surjective.

For any $M \in M_G$ and any $(C_1, \ldots, C_s, Q_1, \ldots, Q_s) \in \psi^{-1}(M)$, it uniquely determines an ordering of edges in M. The number of such orderings that give the same projection G is exactly $|\mathrm{Aut}^*(G)|$, the number of edge automorphisms of G. By Whitney's Theorem [7], for a connected G, which is not K_2 or K_1, $|\mathrm{Aut}^*(G)| = |\mathrm{Aut}(G)|$. We have $|\psi^{-1}(M)| = |\mathrm{Aut}(G)|$.

There are $\binom{n}{|V(G)|}|V(G)|!$ ways to choose $(C_1, \ldots, C_s)$. For $1 \le i \le s$, there are $\binom{d}{d_{v_i}} d_{v_i}!$ ways to choose the queue Q_i. We have

$$|\mathcal{C}| = \binom{n}{|V(G)|}|V(G)|! \prod_{v \in V(G)} \binom{d}{d_v} d_v!.$$

Thus,

$$\begin{aligned} |M_G| &= \frac{|\mathcal{C}|}{|\mathrm{Aut}(G)|} \\ &= \frac{1}{|\mathrm{Aut}(G)|}\binom{n}{|V(G)|}|V(G)|! \prod_{v \in V(G)} \binom{d}{d_v} d_v! \\ &= \frac{f_G(d)}{|\mathrm{Aut}(G)|}\binom{n}{|V(G)|}|V(G)|! d^{|V(G)|}. \end{aligned} \tag{8}$$

For $i \ge 1$, let $\mathcal{G}_i$ be the set of graphs in $\mathcal{G}$ with exactly i edges. Let $\mathcal{M}_i$ be the set of matchings of U whose projection gives a graph $G \in \mathcal{G}_i$; there are *exactly* $|M_G|$ of them, and they are counted in (8). The bad events to be avoided are the projection of some matching from the union $\mathcal{M} = \cup_{i=1}^r \mathcal{M}_i$. For each $M_i \in \mathcal{M}_i$ $(i = 1, 2, \ldots, r)$, we have

$$\Pr(A_{M_i}) = \frac{1}{(nd-1)(nd-3)\cdots(nd-2i+1)}. \tag{9}$$

We have

$$\begin{aligned}
\mu &= \sum_{M\in\mathcal{M}} \Pr(A_M) \\
&= \sum_{i=1}^{r}\sum_{G\in\mathcal{G}_i} \frac{f_G(d)}{|\mathrm{Aut}(G)|}\binom{n}{|V(G)|}|V(G)|!d^{|V(G)|} \\
&\quad \frac{1}{(nd-1)(nd-3)\cdots(nd-2i+1)} \\
&= \sum_{i=1}^{r}\sum_{G\in\mathcal{G}_i} \frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{i-|V(G)|}}\left(1+O\left(\frac{i^2}{n}\right)\right) \\
&= \left(1+O\left(\frac{r^2|\mathcal{G}|}{n}\right)\right)\left(\sum_{G\in\mathcal{G}}\frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}\right). \qquad (10)
\end{aligned}$$

Observe from (10) that $\mu = O(\sum_{G\in\mathcal{G}}\frac{(d-1)^{|E(G)|+\kappa}}{(nd)^\kappa}) = O(\sum_{G\in\mathcal{G}}\frac{(d-1)^{|E(G)|}}{n^\kappa})$. Now we verify the conditions of Theorem 1. Item 1 and 2 are trivial by the definition of r and ϵ. Item 3 can be verified as follows. For two matchings M and M', $A_M \cap A_{M'} = \emptyset$ if any only if M and M' conflict. For $M \in \mathcal{M}_j$, we have

$$\begin{aligned}
\sum_{M':A_{M'}\cap A_M=\emptyset}\Pr(A_{M'}) &= \sum_{i=1}^{r}\sum_{M'\in\mathcal{M}_i:A_{M'}\cap A_M=\emptyset}\Pr(A_{M'}) \\
\text{(by symmetry argument)} &\le \sum_{i=1}^{r}\sum_{M'\in\mathcal{M}_i}\frac{2j}{nd}\Pr(A_{M'}) \\
&\le \frac{2r}{nd}\sum_{i=1}^{r}\sum_{M'\in\mathcal{M}_i}\Pr(A_{M'}) \\
&= \frac{2r}{nd}\mu \\
&< \epsilon. \qquad (11)
\end{aligned}$$

Now we verify item 4. For any $uv \in M \in \mathcal{M}$, we have

$$\begin{aligned}
\sum_{M:uv\in M\in\mathcal{M}}\Pr(A_M) &\le \sum_{i=2}^{r}\sum_{G\in\mathcal{G}_i}\frac{\binom{n}{|V(G)|-2}(|V(G)|-2)!d^{-2}\prod_{v\in V(G)}\binom{d}{d_v}d_v!}{(nd-1)(nd-3)\cdots(nd-2i+1)} \\
&< \sum_{i=2}^{r}\sum_{G\in\mathcal{G}_i}\frac{f_G(d)}{(nd)^{i-|V(G)|+2}}\left(1+O\left(\frac{i^2}{n}\right)\right) \\
&< \epsilon. \qquad (12)
\end{aligned}$$

(We omitted a $\frac{1}{nd-1}$ additive term from the estimate, which was there in [9], as it was there to handle a loop.)

Finally, we have to verify item 5. For any $F \in \mathcal{M}$, we have to estimate $\sum_{M\in\mathcal{M}_F} \Pr_{N-2r}(A_M)$. By the inequality below, this boils down to estimating $\sum_{M\in\mathcal{M}_F} \Pr(A_M) = \sum_{M\in\mathcal{M}_F} \Pr_N(A_M)$, as with $|M| = i$,

$$\begin{aligned} \frac{\Pr_{N-2r}(A_M)}{\Pr_N(A_M)} &\leq \prod_{j=1}^{i} \frac{nd-2j-1}{nd-r-2j-1} \\ &\leq \prod_{j=1}^{i} \left(1 + \frac{2r}{n-2r-2j-1}\right) \leq e^{\frac{2r^2}{nd-4r-1}}. \end{aligned}$$

Assume that $M' \in \mathcal{M}$ intersects F, $M = M' \setminus F \neq \emptyset$, and the projection of M' is a graph $G' \in \mathcal{G}$. Let H be the projection of $F \cap M'$. The graph H is a subgraph of G satisfying $0 < |E(H)| < |E(G)|$. (Otherwise, $G' \subset G$ contradicts to the assumption that $\mathcal{G}$ is balanced.)

We have

$$\begin{aligned} & \sum_{M\in\mathcal{M}_F} \Pr(A_M) \\ \leq\ & \sum_{i=2}^{r} \sum_{G'\in\mathcal{G}_i} \sum_{\substack{H\subset G' \\ E(H)\neq\emptyset}} \frac{\big(n-|V(G)|\big)_{|V(G')|-|V(H)|}(d(d-1))^{(|E(G')|-|E(H)|)}}{\prod_{j=1}^{|E(G')|-|E(H)|}(nd-2j+1)} \\ =\ & \sum_{i=2}^{r} \sum_{G'\in\mathcal{G}_i} \left(1+O\left(\frac{i^2}{n}\right)\right) \sum_{\substack{H\subset G' \\ E(H)\neq\emptyset}} \frac{(d-1)^{|E(G')|-|E(H)|}}{n^{\kappa(G')-\kappa(H)}} \\ =\ & \left(1+O\left(\frac{r^2|\mathcal{G}|}{n}\right)\right) \sum_{G'\in\mathcal{G}} \sum_{\substack{H\subset G' \\ E(H)\neq\emptyset}} \frac{(d-1)^{|E(G')|-|E(H)|}}{n^{\kappa(G')-\kappa(H)}}. \end{aligned}$$

(For the $d(d-1)$ base term in the second line, consider that we can build up G' sequentially by always adding an edge incident to a pre-existing component with at least one edge, starting with the components of H with at least one edge.)

Since G' is balanced, we have $\kappa(G') - \kappa(H) \geq 1$ for any subgraph H with $0 < |E(H)| < |E(G)|$. The last summation can be partitioned into summations over two classes. The first class $\mathcal{C}_1$ consists of H with $\kappa(H) = \kappa(G') - 1$. By Lemma 3, the number of such H is at most $|V(G')|^2$. The second class $\mathcal{C}_2$ consists of H with $\kappa(H) \leq \kappa(G') - 2$; there are most

$2^{|E(G')|}$ of them. We bound $(d-1)^{|E(G')|-|E(H)|}$ by $(d-1)^{|E(G')|-1}$. We have

$$\begin{aligned}\sum_{M\in\mathcal{M}_F}\Pr(A_M) &\leq 2\sum_{G'\in\mathcal{G}}\sum_{\substack{H\subset G'\\ E(H)\neq\emptyset}}\frac{(d-1)^{|E(G')|-|E(H)|}}{n^{\kappa(G')-\kappa(H)}}\\ &\leq 2\sum_{G'\in\mathcal{G}}(d-1)^{|E(G')|-1}\left(\frac{2|V(G')|^2}{n}+\frac{2^{|E(G')|}}{n^2}\right)\\ &< \epsilon.\end{aligned}$$

Finally, the error in (10) does not hurt, as

$$\begin{aligned}e^{-\mu} &= e^{-\left(1+O\left(\frac{r^2|\mathcal{G}|}{n}\right)\right)\left(\sum_{G\in\mathcal{G}}\frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}\right)}\\ &= e^{-\sum_{G\in\mathcal{G}}\frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}}e^{-O\left(\frac{r^2|\mathcal{G}|}{n}\right)\sum_{G\in\mathcal{G}}\frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}}\\ &= e^{-\sum_{G\in\mathcal{G}}\frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}}\left(1-O\left(\frac{r^2|\mathcal{G}|}{n}\right)\sum_{G\in\mathcal{G}}\frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}\right)\\ &= (1-O(\ell))e^{-\sum_{G\in\mathcal{G}}\frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}},\end{aligned}$$

and $e^{-\sum_{G\in\mathcal{G}}\frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}}=(1+O(\ell))e^{-\mu}$. □

Corollary 4.1 *We obtain Theorem 2 from Theorem 4, with the following condition, which is slightly weaker than (6) in [9]:*

$$g^3(d-1)^{2g-3}=o(n) \tag{13}$$

Proof: Note that cycles are exactly the connected balanced graphs with $\kappa=0$. Let C_1 denote the graph of a one-vertex loop and C_2 the graph of a pair of multiedges between two vertices. These are balanced multigraphs with $\kappa=0$. Formally, we did not allow in Theorem 4 balanced multigraphs, however, minor changes in the arguments will allow the inclusion of these two graphs (see in [9] how to handle loops and parallel edges). The formulas extend for C_2 and C_1, if we use as definition $|\mathrm{Aut}(C_2)|=4$ and $f_{C_2}(d)=(d-1)^2$; $|\mathrm{Aut}(C_1)|=2$ and $f_{C_1}(d)=d-1$. Applying Theorem 4 to the family $\mathcal{G}=\mathcal{C}\cup\{C_1,C_2\}$, where $\mathcal{C}\subseteq\{C_3,\ldots,C_{g-1}\}$ for $g\geq 3$ one obtains Theorem 2. □

Corollary 4.2 *Under the conditions of Theorem 4, the probability of obtaining a balanced graph from $\mathcal{G}$ after projection in the configuration model,*

(i) *if* $\sum_{G\in\mathcal{G}} \frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}$ *is separated from zero, is*

$$1-e^{-\sum_{G\in\mathcal{G}} \frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}}+O(\ell)e^{-\sum_{G\in\mathcal{G}} \frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}},$$

(ii) *if* $\sum_{G\in\mathcal{G}} \frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}=o(1)$*, and the first part of (7) is strengthened to* $\frac{r^3}{n}\sum_{G\in\mathcal{G}}(d-1)^{|E(G)|-1}=o(\frac{f_G(d)}{d^{\kappa}|\mathrm{Aut}(G)|})$ *uniformly, is*

$$\sum_{G\in\mathcal{G}} \frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}+O\Big(\ell+(\sum_{G\in\mathcal{G}} \frac{f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}})^2\Big),$$

where ℓ is little-oh of the main term.

Proof: (i) is straightforward. To obtain (ii), use $1-(1+O(\ell))e^x=x+O(\ell+x^2)$ for $\ell=o(1), x=o(1)$. The fact that ℓ is little-oh of the main term follows from the extra assumption in (ii). □

In the *bipartite configuration model* we have two sets, U and V, each containing N mini-vertices, a fixed partition of U into $d_1,...,d_n$ element classes, and a fixed partition of V into $\delta_1,...,\delta_n$ element classes. Any perfect matching between U and V defines a bipartite multigraph with partite sets of size n after a projection contracts every class to single vertex. In the regular case, $d_1=\cdots=d_n=\delta_1=\cdots=\delta_n=d$. We have the following theorem

Theorem 5 *Let $\mathcal{G}$ be a family of connected balanced simple bipartite graphs with non-negative excess κ and $r=\max_{G\in\mathcal{G}}|E(G)|$. In the regular case of the bipartite configuration model, assume $d\geq 3$ and condition (7). Then the probability that the random d-regular multigraph contains no graph in $\mathcal{G}$ is*

$$(1+o(1))e^{-\sum_{G\in\mathcal{G}} \frac{2f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}}.$$

Proof: We outline the proof. For $i=2,\dots,r$, let $\mathcal{M}_i$ be the set of matchings of U and V, whose projection gives a graph $G\in\mathcal{G}$ with i edges. For a fixed $G\in\mathcal{G}$, since G is bipartite, let $n_1(G)$ and $n_2(G)$ be the size of vertex partition classes. The number of matchings, whose projection is G, is exactly

$$\frac{2}{|\mathrm{Aut}(G)|}\binom{n}{n_1(G)}n_1(G)!\binom{n}{n_2(G)}n_2(G)!\prod_{v\in V(G)}\binom{d}{d_v}d_v!.$$

This formula is similar to Equation (8). If G has no automorphism switching its two colorclasses (in particular when $n_1\neq n_2$), then we can select

$n_1(G)$ classes from the n classes of U and select $n_2(G)$ classes from the n classes of V, or vice versa. This explains the constant factor 2. If G has an automorphism switching its two colorclasses, then selecting $n_1 = n_2$ classes from U and V, we obtain each copy of G $|Aut^*(G)|/2$ times from matchings. The bad events correspond to a matching from the union $\mathcal{M} = \cup_{i=1}^r \mathcal{M}_i$. For each $M_i \in \mathcal{M}_i$ $(i = 1, 2, \ldots, r)$, we have

$$\Pr(A_{M_i}) = \frac{(dn-2i)!}{(dn)!}. \tag{14}$$

We have

$$\begin{aligned}
&\sum_{M \in \mathcal{M}} \Pr(A_M) \\
= &\sum_{i=2}^{r} \sum_{G \in \mathcal{G}_i} \frac{2}{|\mathrm{Aut}(G)|} (n)_{n_1(G)} (n)_{n_2(G)} \prod_{v \in V(G)} \binom{d}{d_v} d_v! \frac{(dn-2i)!}{(dn)!} \\
= &\left(1 + O\left(\frac{r^2|\mathcal{G}|}{n}\right)\right) \sum_{G \in \mathcal{G}} \frac{2 f_G(d)}{|\mathrm{Aut}(G)|(nd)^{\kappa(G)}}. \qquad (15)
\end{aligned}$$

All the estimates go through as in the proof of Theorem 2. □

Applying Theorem 5 to a family

$$\mathcal{G} = \{C_2\} \cup \mathcal{C} \quad \text{with} \quad \mathcal{C} \subseteq \{C_4, C_6, \ldots, C_{2g-2}\}$$

(a slight extension to include C_2, like in [9]), we get another theorem of McKay, Wormald and Wysocka [12], who actually had it without g^3 in (16). [9] reproved this theorem with g^6 in the condition using Theorem 1.

Theorem 6 *In the regular case of the bipartite configuration model, assume that g is even, $d \geq 3$, and*

$$g^3(d-1)^{2g-3} = o(n). \tag{16}$$

Then the probability that the random bipartite d-regular multigraph does not contain a cycle of length $\mathcal{C} \subseteq \{2, 4, 6, ..., g-2\}$, is

$$(1 + o(1))e^{-\sum_{i \in \mathcal{C}} \frac{(d-1)^i}{i}}.$$

Corollary 6.1 *Corollary 4.2 applies to the bipartite regular configuration model, changing $f_G(d)$ to $2f_G(d)$.*

We are left with an open problem of finding asymptotics for the occurrence of an element of $\mathcal{G}$ and obtaining a *simple* multigraph simultaneously.

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