Some Applications of Spanning Trees in $K_{s,t}$

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Abstract

We partition the set of spanning trees contained in the complete graph K_n into spanning trees contained in the complete bipartite graph $K_{s,t}$. This classification shows that some properties of spanning trees in K_n can be derived from trees in $K_{s,t}$. We use Abel's binomial theorem and the formula for spanning trees in $K_{s,t}$ to obtain a proof of Cayley's theorem using partial derivatives. Some results concerning non-isomorphic spanning trees are presented. In particular we count these trees for Q_3 and the Petersen graph.

Keywords. Abel's binomial theorem, Cayley's theorem, hypercube, Petersen graph, spanning trees

1 Introduction

We use the standard notation and terminology which can be found, e.g., in [12]. Let $\tau(G)$ denote the number of labelled spanning trees in a graph G. Let K_n denote the complete graph of n vertices and $K_{s,t}$ the complete bipartite graph with partite sets containing s and t vertices, respectively. It is well known, as in e.g. [2, 3, 4, 5, 6, 10] that

$$\tau(K_n) = n^{n-2}, \qquad n \geq 2 \tag{1}$$

$$\tau(K_{s,t}) = s^{t-1}t^{s-1}, \qquad s,t \geq 1. \tag{2}$$

We remark that (1) is often referred to as Cayley's theorem. Let $s + t = n$, where $1 \leq s \leq t$. We have the following observation:

Theorem 1. *With $n \geq 2$, any spanning tree T in K_n is a spanning tree in $K_{s,t}$ for a unique pair (s,t), with $1 \leq s \leq t$ and $s + t = n$.*

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Proof. Consider a spanning tree T in K_n. Because T is a connected bipartite graph it is uniquely 2-colorable. So construct this unique bipartition by properly 2-coloring the vertex set of T with colors red (R) and blue (B). Let the number of red vertices be s and the number of blue vertices be t; w.l.o.g., let $s \leq t$. We then have T is a spanning tree in this $K_{s,t}$. □

The converse is straightforward.

Theorem 2. *With $s + t = n$, any spanning tree in $K_{s,t}$ is a spanning tree in K_n.*

Proof. This follows since $K_{s,t}$ is a spanning subgraph of K_n. □

Theorem 3.

$$\sum_{s=1}^{n-1} \binom{n}{s} \tau(K_{s,n-s}) = 2\tau(K_n).$$

Proof. By combining Theorems 1 and 2 we see that to find $\tau(K_n)$ we can enumerate all labelled spanning trees in the possible $K_{s,t}$ graphs. The double count occurs from the 2-colorings of the partite sets. □

We now proceed to show the LHS of Theorem 3 implies the RHS yielding a calculus based proof of Cayley's theorem. We shall apply Abel's binomial formula, [1], which states that for any x, y, and z that:

$$(x+y)^n = \sum_{s=0}^{n} \binom{n}{s} x(x-sz)^{s-1}(y+sz)^{n-s}. \tag{3}$$

Theorem 4. $\tau(K_{s,t}) \implies \tau(K_n)$. *In words, the formula for $\tau(K_n)$ can be derived from the formula for $\tau(K_{s,t})$.*

Proof. From (3) we have

$$n(x+y)^{n-1} = \frac{\partial}{\partial x}(x+y)^n = \sum_{s=0}^{n} \binom{n}{s} s(x-z)(x-sz)^{s-2}(y+sz)^{n-s} \tag{4}$$

and consequently,

$$\begin{aligned} \frac{\partial^2}{\partial y \partial x} &= n(n-1)(x+y)^{n-2} \\ &= \sum_{s=0}^{n} \binom{n}{s} s(x-z)(x-sz)^{s-2}(n-s)(y+sz)^{n-s-1}. \end{aligned} \tag{5}$$

We also have,

$$n(x+y)^{n-1} = \frac{\partial}{\partial y}(x+y)^n = \sum_{s=0}^{n} \binom{n}{s} x(x-sz)^{s-1}(n-s)(y+sz)^{n-s-1}. \tag{6}$$

By substituting $x = n$, $y = 0$, and $z = 1$ into (5) and (6), we obtain, respectively, (7) and (8):

$$n^{n-1} = \sum_{s=1}^{n} \binom{n}{s} s^{n-s}(n-s)^{s-1} \tag{7}$$

$$n^{n-1} = \sum_{s=1}^{n} \binom{n}{s} (n-s)^{s} s^{n-s-1}. \tag{8}$$

Adding (7) to (8) gives

$$2n^{n-1} = \sum_{s=1}^{n} \binom{n}{s} s^{n-s-1}(n-s)^{s-1} n, \tag{9}$$

which yields the equation in Theorem 3. This gives a proof of Cayley's theorem using partial derivatives. □

The identity in (9) can also be found in [2, 8, 11]. The ideas in Theorems 1 and 2 are also valid when graphs are unlabelled, since the unique bipartition aspect is a structural property of the graph T. So, for a connected graph G, let $I(G)$ be the number of non-isomorphic spanning trees in G. We have:

Theorem 5. $I(K_n) = \sum_{s=1}^{\lfloor n/2 \rfloor} I(K_{s,n-s})$. □

A formula for $I(K_{s,t})$ would then give a formula for $I(K_n)$. We wrote a computer program that generates the set of labelled spanning trees in a graph G. It then partitions this set into its isomorphism classes. Table 1 gives some results found when $G = K_{s,t}$, $I(K_{5,5})$ being the largest calculation in terms of computing time we have been able to produce. The top number in row s and column t corresponds to $\tau(K_{s,t})$ and the bottom number is $I(K_{s,t})$. We have not seen these numbers in Table 1 in the literature.

Observational examples of Theorem 5 and Table 1, using well known values of some $I(K_n)$, are:

$$\begin{aligned} I(K_6) = 6 &= I(K_{1,5}) + I(K_{2,4}) + I(K_{3,3}) \\ &= 1 + 2 + 3, \\ I(K_7) = 11 &= I(K_{1,6}) + I(K_{2,3}) + I(K_{3,4}) \\ &= 1 + 3 + 7, \end{aligned}$$

and

$$\begin{aligned} I(K_{10}) = 106 &= I(K_{1,9}) + I(K_{2,8}) + I(K_{3,7}) + I(K_{4,6}) + I(K_{5,5}) \\ &= 1 + 4 + 19 + 45 + 37. \end{aligned}$$

We would like to derive a general or asymptotic formula for $I(K_{s,t})$. An asymptotic formula for $I(K_n)$ is given by Otter [9]

$$I(K_n) \sim pn^{-5/2} r^{-n}, \quad \text{where } p \text{ and } r \text{ are constants.}$$

	1	2	3	4	5	6	7	8	...	n
1	1 1	1 1	1 1	1 1	1 1	1 1	1 1	1 1		1 1
2	...	4 1	12 2	32 2	80 3	192 3	448 4	1042 4		
3	...	...	81 3	432 7	2025 10	8748 14	35721 19	139968 24		
4	...	...	...	4096 9	32000 28	221184 45				
5	...	...	...	...	390625 37					

Table 1: Values of τ and I for $K_{s,t}$

Our work so far has given partition numbers for $I(K_{2,n})$ and $I(K_{3,n})$. Let $p_k(n)$ denote the number of partitions of an integer n into k or fewer parts. Then, we have:

$$I(K_{2,n}) = p_2(n-1) = \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \quad \text{for } n \geq 2 \tag{10}$$

$$I(K_{3,n}) = p_3(n-1) + \sum_{k=0}^{n-2} p_2(n-2-k), \quad \text{for } n \geq 4. \tag{11}$$

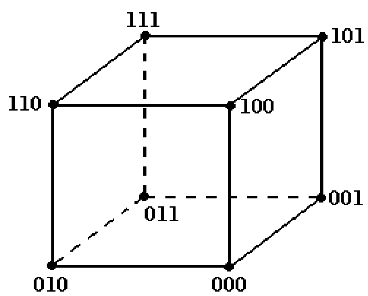
In (11), we adopt the convention that $p_2(0) = 1$. As an example of (11),

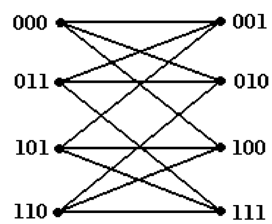
$$\begin{aligned} I(K_{3,5}) &= 10 = p_3(4) + p_2(3) + p_2(2) + p_2(1) + p_2(0) \\ &= 4 + 2 + 2 + 1 + 1. \end{aligned}$$

We ran our tree isomorphism program on some other popular graphs, namely Q_3 and the Petersen graph. Let Q_n denote the n-dimensional cube and let P denote the Petersen graph. $\tau(Q_n)$ is known, the values $\tau(Q_3) = 384$ and $\tau(P) = 2000$ are generally known, however, it appears that $I(Q_n)$ and $I(P)$ may not be so universally known. After applying our algorithm to Q_3 and P, we have found that $I(Q_3) = 6$ and $I(P) = 20$. Table 2 gives the breakdown of the size of each isomorphism class in Q_3. For example, row 2 denotes that there are 3 classes, each containing 48 trees. Table 3 gives drawings of a representative tree from each of the 6-classes. We remark the class containing the spanning paths has 72 trees. Table 4 gives the different class sizes for the Petersen graph. On Austin Mohr's website [7], there are drawings of representative trees for the Petersen graph similar to Table 3. There are also drawings for the trees given in Table 1.

$T = 384$

$I = 6$



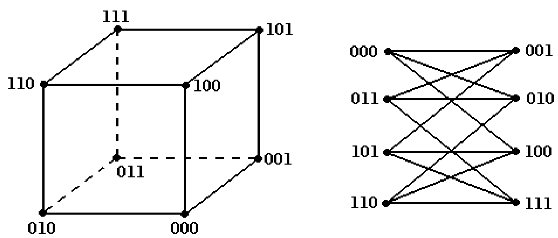


Distribution of Class Sizes

	Num Trees	Size of Class
	24	1
	48	3
	72	1
	144	1
Total	384	

Table 2: Q_3

$$T = 384$$
$$I = 6$$



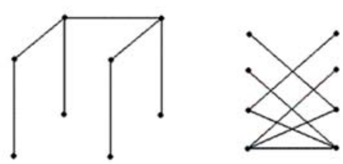
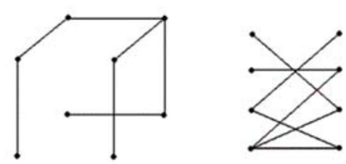
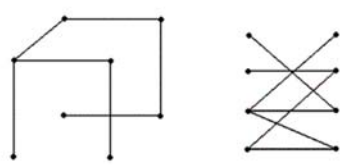
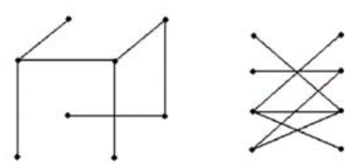
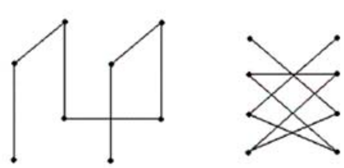
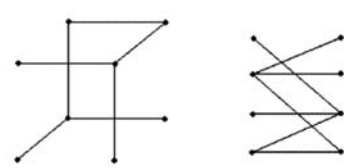
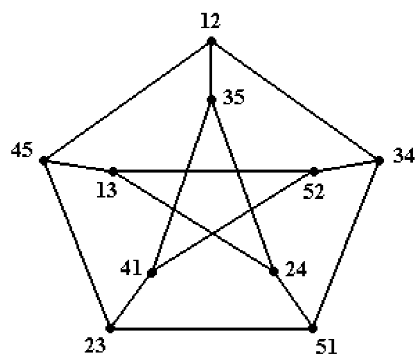
48 in class	48 in class
	
144 in class	**48 in class**
	
72 in class	**24 in class**
	

Table 3: Q_3

$$T = 2000$$
$$I = 20$$



Distribution of Class Sizes

	Num Trees	Size of Class
	10	1
	30	1
	40	1
	60	4
	120	12
	240	1
Total	2000	20

Table 4: Petersen Graph

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