Math 774 Homework 5

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Problem 1

Proposition 1. *For an arbitrary number of colors t and for $r = 2$,*

$$R(k_1, k_2, \ldots, k_t) \leq 2 + \sum_{i=1}^{t} (R(k_1, k_2, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_t) - 1).$$

Proof. Let $n = 2 + \sum_{i=1}^{t} (R(k_1, k_2, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_t) - 1)$ and consider any coloring of the edges of K_n. Select an arbitrary vertex u and let A_i denote the set of all vertices v such that the edge uv is colored i. Observe that,

$$\sum_{i=1}^{t} |A_i| = 1 + \sum_{i=1}^{t} (R(k_1, k_2, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_t) - 1).$$

Thus, by the pigeonhole principle, there exists j such that

$$|A_j| \geq R(k_1, k_2, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_t).$$

By definition of the Ramsey number, A_j contains a monochromatic K_{k_i} colored i for $i \neq j$ or A_j contains a monochromatic $K_{k_j - 1}$ colored j. In the former case, we are done (we have found one of the desired subgraphs). In the latter case, recall that any edge from u into A_j is colored j. Since u sees every vertex in A_j, it follows that $\{u\} \cup A_j$ contains a monochromatic K_{k_j}. □

Lemma 1. *For $k \geq 2$,*

$$\text{frac}((k-1)!e) > \frac{1}{k}.$$

Proof. From the series expansion of e, we have

$$
\begin{aligned}
(k-1)!e &= (k-1)! \sum_{n=0}^{\infty} \frac{1}{n!} \\
&= (k-1)! \left(\sum_{n=0}^{k-1} \frac{1}{n!} + \sum_{n=k}^{\infty} \frac{1}{n!} \right) \\
&= \sum_{n=0}^{k-1} \frac{(k-1)!}{n!} + \sum_{n=k}^{\infty} \frac{(k-1)!}{n!}.
\end{aligned}
$$

Observe that every term in the left summation is an integer, and so the sum is an integer. We next bound the right summation by 1.

$$
\begin{aligned}
\sum_{n=k}^{\infty} \frac{(k-1)!}{n!} &= \frac{(k-1)!}{k!} + \frac{(k-1)!}{(k+1)!} + \cdots \\
&= \frac{1}{k} + \frac{1}{k(k+1)} + \cdots \\
&< \sum_{n=1}^{\infty} \frac{1}{k^n} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= 1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\operatorname{frac}((k-1)!e) &= \sum_{n=k}^{\infty} \frac{(k-1)!}{n!} \\
&> \frac{1}{k}.
\end{aligned}
$$

□

Corollary 1. *Letting $k_i = 3$ in the above for all i, the Ramsey number*

$$R(3, \ldots, 3) \leq 1 + t! \cdot e,$$

where e denotes Euler's number.

Proof. We establish the first two cases directly.

For $t = 1$, we have $R(3) = 3$ (with only a single color available, K_3 is always monochromatic). Thus,

$$\begin{aligned} R(3) &= 3 \\ &\leq 1 + 1!e. \end{aligned}$$

Before continuing with the induction, notice that if $k_i = 2$ for some i, then the color i cannot be used (coloring even a single edge results in a monochromatic K_2). Thus, for example,

$$R(\underbrace{2, 3, \ldots, 3}_{t \text{ entries}}) = R(\underbrace{3, \ldots, 3}_{t-1 \text{ entries}}).$$

This result holds regardless of which k_i is set as 2.

Suppose now that the claim holds for $t = k-1$. Via the lemma, we make use of the slightly stronger inductive hypothesis

$$\begin{aligned} R(\underbrace{3, \ldots, 3}_{k-1 \text{ entries}}) &\leq \lfloor 1 + (k-1)!e \rfloor \\ &< 1 + (k-1)!e - \frac{1}{k}. \end{aligned}$$

Now, using the bound in the previous problem, we have

$$\begin{aligned} R(\underbrace{3, \ldots, 3}_{k \text{ entries}}) &\leq 2 + \sum_{i=1}^{k} (R(\underbrace{3, \ldots, 3}_{k-1 \text{ entries}}) - 1) \\ &= 2 + k(R(\underbrace{3, \ldots, 3}_{k-1 \text{ entries}}) - 1) \\ &< 2 + k\left(1 + (k-1)!e - \frac{1}{k} - 1\right) \\ &= 1 + k!e. \end{aligned}$$

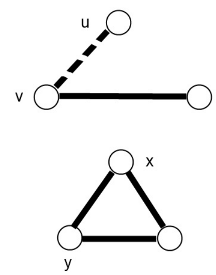
□

Problem 2

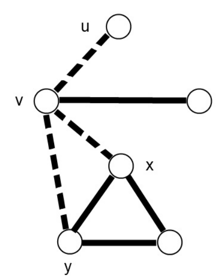
Proposition 2. *Any 2-coloring of the edges of K_6 contains at least two (not necessarily disjoint) monochromatic triangles.*

Proof. We will color the edges of K_6 "solid" and "dashed". By Ramsey's theorem, we are assured one monochromatic triangle (without loss of generality, let it be solid). We now attempt to color the remaining edges to avoid another monochromatic triangle, but this endeavor will ultimately fail.

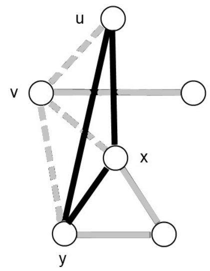
Among the three vertices not beloning to the aforementioned solid triangle, at least one edge is solid and one dashed (else there is another monochromatic triangle).



Observe next that v may not have two solid edges to vertices of the solid triangle (else there is another mono-solid triangle). Thus, at least two of these edges are dashed.



Now, the edges ux and uy must both be solid (else $uvxu$ or $uvyu$ is a mono-dashed triangle), but this forms the mono-solid triangle $uxyu$.



∎

Problem 3

Theorem 1. *(Schur) For all k there exists n (depending on k) such that if $[n]$ is k-colored, then there exists a monochromatic solution $x, y, z \in [n]$ to the equation $x + y = z$.*

Proof. Let k be given and let $c : [n] \to [k]$ be any k-coloring of $[n]$ (the value of n will be determined later). Define now $f : \binom{[n]}{2} \to [k]$ via $f(\{i,j\}) = c(|i-j|)$. By Ramsey, we will take n to be large enough so that f admits a monochromatic 3-subset $\{a,b,c\}$ of $\binom{[n]}{2}$. Without loss of generality, suppose $a < b < c$. Let now $x = b - a$, $y = c - b$, and $z = c - a$. Since $\{a,b,c\}$ is monochromatic under f, we have $f(\{a,b\}) = f(\{b,c\}) = f(\{a,c\})$, but this is precisely $c(|b-a|) = c(|b-c|) = c(|a-c|)$. Hence, $c(x) = c(y) = c(z)$, and so x, y, and z indeed belong to the same subset of $[n]$ under c. Finally,

$$\begin{aligned} x + y &= (b-a) + (c-b) \\ &= c - a \\ &= z, \end{aligned}$$

as desired. ∎

Problem 4

Proposition 3. *For all k and n satisfying $n \geq k$ and $\binom{n}{k} < 2^{\binom{k}{2}-1}$,*

$$R(k,k) > n.$$

Proof. Let n and k be given as described above and let $A(n,k)$ denote the number of colorings for which a fixed selection of k vertices induces a monochromatic K_k. The $\binom{k}{2}$ edges induced by the k vertices must be either all red or all blue. The remaining $\binom{n}{2} - \binom{k}{2}$ edges may receive any color. Hence,

$$\begin{aligned} A(n,k) &= 2 \cdot 2^{\binom{n}{2}-\binom{k}{2}} \\ &= 2^{\binom{n}{2}-\binom{k}{2}+1}. \end{aligned}$$

Now, we may select the k vertices to induce the monochromatic K_k in $\binom{n}{k}$ ways. Hence, the number of two-colorings of the edges of K_n that contain a monochromatic K_k is at most

$$\begin{aligned} \binom{n}{k} A(n,k) &< 2^{\binom{k}{2}-1} \cdot 2^{\binom{n}{2}-\binom{k}{2}+1} \\ &= 2^{\binom{n}{2}}. \end{aligned}$$

As there are a total of $2^{\binom{n}{2}}$ two-colorings of the edges of K_n, we conclude that the probability of choosing such a coloring uniformly at random is at most

$$\begin{aligned} \frac{\binom{n}{k} A(n,k)}{2^{\binom{n}{2}}} &< \frac{2^{\binom{n}{2}}}{2^{\binom{n}{2}}} \\ &= 1. \end{aligned}$$

In other words, the probability of avoiding a monochromatic K_k is nonzero, and so $R(k,k) > n$. $\square$

Lemma 2. *For all $n \geq k$,*

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k \frac{1}{\sqrt{2\pi k}}(1+o(1)) \text{ as } k \to \infty.$$

Proof. For all $n \geq k$,

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-k)!} \frac{1}{k!} \\ &\leq n^k \frac{1}{k!} \end{aligned}$$

$$= n^k k^{-k} e^k \frac{1}{\sqrt{2\pi k}}(1 + o(1))$$
$$= \left(\frac{ne}{k}\right)^k \frac{1}{\sqrt{2\pi k}}(1 + o(1)).$$

□

Proposition 4. *For all k,*

$$R(k,k) \geq c_1 k 2^{\frac{k}{2}}(1 + o(1)) \text{ as } k \to \infty$$

for some constant c_1.

Proof. From the proposition, we know that the probability of choosing a random coloring containing a monochromatic K_k is at most (as $k \to \infty$)

$$\frac{\binom{n}{k} A(n,k)}{2^{\frac{n}{2}}} \leq \frac{\left(\frac{ne}{k}\right)^k \frac{1}{\sqrt{2\pi k}} 2^{\binom{n}{2} - \binom{k}{2} + 1}}{2^{\binom{n}{2}}}$$
$$= \left(\frac{ne}{k}\right)^k \frac{1}{\sqrt{2\pi k}} 2^{-\binom{k}{2}+1}.$$

Now, substituting $c_1 k 2^{\frac{k}{2}}$ for n,

$$\frac{\binom{n}{k} A(n,k)}{2^{\frac{n}{2}}} \leq \left(\frac{c_1 k 2^{\frac{k}{2}} e}{k}\right)^k \frac{1}{\sqrt{2\pi k}} 2^{-\binom{k}{2}+1}$$
$$= 2^{\frac{k^2}{2}} (c_1 e)^k \frac{2}{\sqrt{2\pi k}} 2^{-\binom{k}{2}}.$$

Taking $c_1 = \frac{1}{2^{\frac{1}{2}} e}$, we have

$$\frac{\binom{n}{k} A(n,k)}{2^{\frac{n}{2}}} \leq 2^{\frac{k^2}{2}} \left(\frac{1}{2^{\frac{1}{2}} e} \cdot e\right)^k \frac{2}{\sqrt{2\pi k}} 2^{-\binom{k}{2}}$$
$$= 2^{\frac{k^2}{2}} \frac{2}{2^{\frac{k}{2}} \sqrt{2\pi k}} 2^{-\frac{k^2}{2} + \frac{k}{2}}$$
$$= \frac{2}{\sqrt{2\pi k}}$$
$$\to 0.$$

as $k \to \infty$. Hence, for large k, there is a nonzero probability of avoiding a monochromatic K_k in a coloring of K_n whenever $n \geq c_1 k 2^{\frac{k}{2}}$. Therefore, $R(k,k) \geq c_1 k 2^{\frac{k}{2}}(1 + o(1))$. □

Proposition 5. *For all k,*

$$R(k,k) \leq c_2 \frac{1}{\sqrt{k}} 4^k (1+o(1)) \text{ as } k \to \infty$$

for some constant c_2.

Proof. Using the fact that $R(k,l) \leq \binom{k+l-2}{k-1}$, we have

$$R(k,k) \leq \binom{2k-2}{k-1}.$$

By a previous result on the middle binomial coefficients, this gives, as $k \to \infty$,

$$\begin{aligned} R(k,k) &= \sqrt{\frac{2}{\pi(2k-2)}} 2^{2k-2} \\ &= \frac{1}{4\sqrt{\pi(k-1)}} 4^k \\ &\leq \frac{1}{4\sqrt{\pi}} \frac{1}{\sqrt{k}} 4^k (1+o(1)). \end{aligned}$$

Taking $c_2 = \frac{1}{4\sqrt{\pi}}$ gives the desired result. □

Proposition 6. *As $k \to \infty$,*

$$\frac{1}{2}(1+o(1)) \leq \frac{\log_2 R(k,k)}{k} \leq 2(1+o(1)).$$

Proof. From the previous results, we have, as $k \to \infty$

$$c_1 k 2^{\frac{k}{2}} (1+o(1)) \leq R(k,k) \leq c_2 \frac{1}{\sqrt{k}} 4^k (1+o(1)).$$

Since $\log_2(1+o(1)) \to 0$, we can disregard this factor in the following. Now,

$$\begin{aligned} \frac{\log_2\left(c_1 k 2^{\frac{k}{2}}\right)}{k} &\leq \frac{\log_2(R(k,k))}{k} \leq \frac{\log_2\left(c_2 \frac{1}{\sqrt{k}} 4^k\right)}{k} \\ \frac{\log_2(c_1 k)}{k} + \frac{\log_2\left(2^{\frac{k}{2}}\right)}{k} &\leq \frac{\log_2(R(k,k))}{k} \leq \frac{\log_2\left(c_2 \frac{1}{\sqrt{k}}\right)}{k} + \frac{\log_2\left(4^k\right)}{k} \\ \frac{\log_2(c_1 k)}{k} + \frac{\frac{k}{2}\log_2(2)}{k} &\leq \frac{\log_2(R(k,k))}{k} \leq \frac{\log_2\left(c_2 \frac{1}{\sqrt{k}}\right)}{k} + \frac{k\log_2(4)}{k} \end{aligned}$$

$$
\begin{aligned}
\frac{\log_2(c_1 k)}{k} + \frac{1}{2} \quad &\leq \tfrac{\log_2(R(k,k))}{k} \quad \leq \frac{\log_2\left(c_2 \frac{1}{\sqrt{k}}\right)}{k} + 2 \\
\frac{1}{2}(1 + o(1)) \quad &\leq \tfrac{\log_2(R(k,k))}{k} \quad \leq 2(1 + o(1)).
\end{aligned}
$$

□

Problem 5

Theorem 2. *(Erdős) For all n, the maximum size of a subset of $[n]$ containing no pair of distinct i and j with i dividing j is*

$$\left\lfloor \frac{n+1}{2} \right\rfloor.$$

Proof. Let $k = \left\lfloor \frac{n}{2} \right\rfloor$ and let A be the set $\{k+1, k+2, \ldots, n\}$. For all $i, j \in A$ with $i < j$,

$$
\begin{aligned}
1 &< \frac{j}{i} \\
&\leq \frac{n}{k+1} \\
&< 2,
\end{aligned}
$$

and so i does not divide j. Moreover,

$$
\begin{aligned}
|A| &= n - (k+1) + 1 \\
&= n - k \\
&= \left\lfloor \frac{n+1}{2} \right\rfloor.
\end{aligned}
$$

It remains to show that there is no larger subset satisfying the desired divisibility property. To see that this is the case, partition $[n]$ into sets B_r where

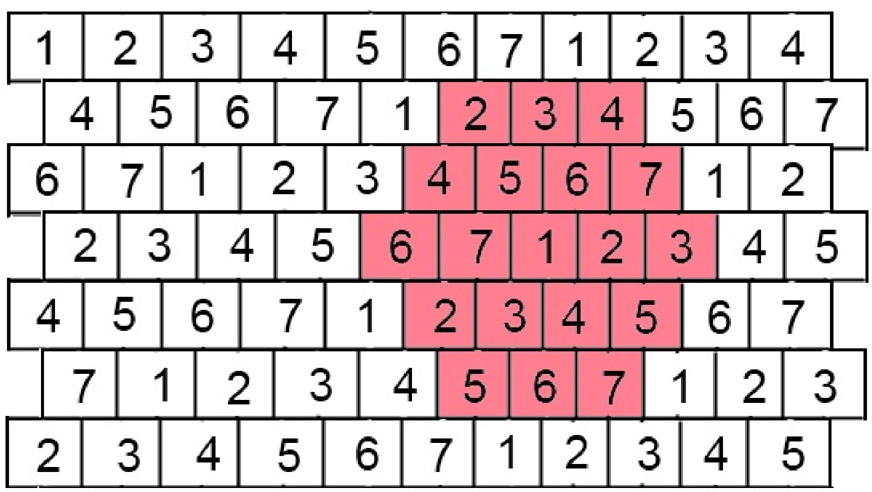
$$B_r = \{2^m r \in [n] \mid r \text{ is odd}\}.$$

By the Fundamental Theorem of Arithmetic, the B_r indeed partition $[n]$. Moreover, there are $\left\lfloor \frac{n+1}{2} \right\rfloor$ such sets (one for each odd number belonging to $[n]$). Thus, by the Pigeonhole Principle, any subset of $[n]$ having more than $\left\lfloor \frac{n+1}{2} \right\rfloor$ elements contains two elements that belong to the same B_{r_0}. That is, any such subset of $[n]$ contains two elements of the form $2^m r_0$ and $2^{m'} r_0$ (without loss of generality, let $m < m'$). Since $2^m r_0$ divides $2^{m'} r_0$, $\left\lfloor \frac{n+1}{2} \right\rfloor$ is thus an attainable upper bound on the size of a subset satisfying the divisibility property. □

Problem 6

Proposition 7. *Seven colors are sufficient to color the points of $\mathbb{R}^2$ so that no two points exactly one unit apart are the same color.*

Proof. In the figure below, tile the plane with a staggered grid. Each of the blocks has diagonal length 1. So there is no overlap, the left and bottom edges of a block are included, while the right and top edges are not. We the blocks in a given row consecutively with the integers 1 through 7. When moving to the next row, we shift the coloring so that, for example, the boxes colored 1 in each row are just out of reach of each other (that is, the distance between them is as small as possible, but still greater than 1). Contintuing this pattern, we can inspect the image to conclude that no two boxes with the same color are within distance one of each other. As an example, the figure below shows highlighted all boxes within distance one of the central box colored 1. Observe that no other highlighted box receives the color 1.



□

Proposition 8. *At least four colors are necessary to color a finite set of non-overlapping pennies in the plane such that pennies that touch each other are colored differently. (By considering unit diameter pennies and coloring their centers, it follows that at least four colors are necessary to color the unit distance graph of $\mathbb{R}^2$.)*

Proof. In the figure below, we assign (without loss of generality) the color green to the penny labeled "start". The two pennies counterclockwise of “start” must receive both the colors blue and red (it makes no difference which penny receives which color), since they both touch “start”. The single penny counterclockwise of these two must, for the same reason, receive the color green. Proceeding in this way, we reach the final penny which has no legal color choice. Therefore, at least four colors are required.

□

Proposition 9. *Four colors suffice to color any finite set of non-overlapping pennies in the plane such that pennies that touch each other are colored differently.*

Proof. Define the slope between two pennies to be the slope between their centers. Since there are a finite number of pennies, there are only a finite number of possible slopes between them (at most $\binom{n}{2}$ for n pennies). Choose any slope m not appearing in the collection of all slopes between pennies. Imagine a line with slope m and y-intercept sufficiently small so that it is below all pennies. If we gradually increase the y-intercept, the line will see the centers of pennies one at a time (if not, then some two pennies have slope m, which is impossible). Thus, the procedure of sliding the line upward gives a total ordering of the pennies. In this ordering, a penny is preceeded by at most three of its neighbors, since a penny can have at most six neighbors and this occurs only when they encircle it (since they are non-overlapping). Hence, a greedy coloring (at each step in the ordering, use the smallest legal color) suffices to color the pennies using at most four colors. $\square$