University of South Carolina
Math 222: Math for Elementary Educators II
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Section 002
Fall 2010

Quiz 3 Solutions

1. Astronomers use light-years to measure distances between celestial bodies. A light-year is the distance light travels in 1 year. The speed of light is (roughly) 300,000 km/sec.

1a. How long is 1 light-year in kilometers?

Solution: Light travels 300,000 kilometers every second, so we need to turn that speed into kilometers per year.

$$\frac{300,000 \text{ km}}{1 \text{ sec}} \cdot \frac{60 \text{ sec}}{1 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ hour}} \cdot \frac{24 \text{ hour}}{1 \text{ day}} \cdot \frac{365 \text{ day}}{1 \text{ year}}$$

When we multiply it all together, all the units except for kilometers on top and years on bottom will cancel (so we know we've set up the solution correctly). All this simplifies to

$$\frac{9.5 \cdot 10^{12} \text{ km}}{1 \text{ year}}.$$

So, a lightyear (the distance light travels in a year) is $9.5 \cdot 10^{12}$ kilometers.

1b. The nearest star (other than the sun) is Alpha Centauri. It is 4.34 light-years from Earth. How far is that in kilometers?

Solution: We know from part a how long a single lightyear is, so we just multiply that answer by 4.34 to get $4.1 \cdot 10^{13}$ kilometers.

1c. How many years will it take a rocket traveling 60,000 km/hr to reach Alpha Centauri?

Solution: We use the fact that distance equals speed times travel time. In other words, travel time equals distance divided by speed. So, we get

$$\begin{aligned} \text{time} &= \frac{4.1 \cdot 10^{13} \text{ km}}{60000 \frac{\text{km}}{\text{hr}}} \\ &= 684331200 \text{ hr}. \end{aligned}$$

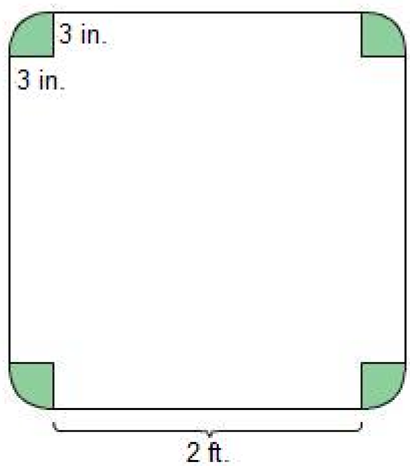
So now we know how many hours it takes, but we want to convert this to years.

$$\frac{684331200 \text{ hr}}{1} \cdot \frac{1 \text{ day}}{24 \text{ hr}} \cdot \frac{1 \text{ year}}{365 \text{ days}}$$

which comes out to 78,120 years.

1d. This question was omitted.

2. The figure below is a square with its corners rounded into perfect quarter-circles. Give your answers to the following in inches, square inches, or cubic inches, as appropriate.



2a. Find the perimeter of the shape.

Solution: The key to these kinds of problems is breaking the shape up into pieces you know about. This perimeter can be viewed as four 2-foot line segments and four quarter-circles of radius 3 inches. So, our preliminary formula might look like

$$\text{perimeter} = 4 \cdot \text{line segment} + 4 \cdot \text{quarter-circle}.$$

Since our answers are supposed to be in inches, we should convert the 2-foot line segment into 24 inches.

The perimeter (usually called circumference) of a circle is given by $2\pi r$, which is 6π inches in this case. Since we are only interested in *quarter*-circles, we divide this by 4 to get $\frac{3\pi}{2}$ inches.

Returning to our formula, we have

$$\begin{aligned}\text{perimeter} &= 4(24 \text{ inches}) + 4(\frac{3\pi}{2} \text{ inches})\\ &= 96 + 6\pi \text{ inches}\end{aligned}$$

2b. Find the area of the shape.

Solution: Here, we might break up the shape into one central square, four rectangles, and four quarter-circles (a picture will help you see the breakdown). We might write this as

$$\text{area} = 1 \cdot \text{square} + 4 \cdot \text{rectangle} + 4 \cdot \text{quarter-circle}.$$

The square is 24 inches by 24 inches, so its area is 576 square inches.

The rectangles are each 3 inches by 24 inches, so each area is 72 square inches.

The area of a circle of radius r is πr^2, which is 9π in this case. We are only interested in *quarter*-circles, so we divide this by 4 to get $\frac{9\pi}{4}$ square inches per quarter-circle.

Returning to our formula, we have

$$\begin{aligned}\text{area} &= 1 \cdot \text{square} + 4 \cdot \text{rectangle} + 4 \cdot \text{quarter-circle}\\ &= 1 \cdot (576 \text{ square inches}) + 4 \cdot (72 \text{ square inches}) + 4 \cdot (\frac{9\pi}{4} \text{ square inches})\\ &= 576 \text{ square inches} + 288 \text{ square inches} + 9\pi \text{ square inches}\\ &= 864 + 9\pi \text{ square inches.}\end{aligned}$$

2c. Find the volume of a 6 inch tall "cylinder" having this shape as its bases.

Solution: The volume of a cylinder is given by the area of its base times the height of the cylinder. We found the area of the base in part b, and we know the height.

$$\begin{aligned}\text{volume} &= \text{area of base} \cdot \text{height of cylinder}\\ &= (864 + 9\pi \text{ square inches}) \cdot (6 \text{ inches})\\ &= 5184 + 54\pi \text{ cubic inches.}\end{aligned}$$

2d. Find the volume of a 6 inch tall "cone" having this shape as its base.

Solution: The volume of a cone is one-third the volume of the related cylinder (by "related", I mean they have exactly the same base).

$$\begin{aligned}\text{volume} &= \frac{1}{3}(\text{volume of related cylinder})\\ &= \frac{1}{3}(5184 + 54\pi \text{ cubic inches})\\ &= 1728 + 18\pi \text{ cubic inches.}\end{aligned}$$

2e. Find the surface area of the "cylinder" in part c.

Solution: The surface area can be viewed as two bases plus the lateral face, which is a rectangle. This means

$$\text{surface area} = 2 \cdot (\text{bases}) + 1 \cdot (\text{lateral face}).$$

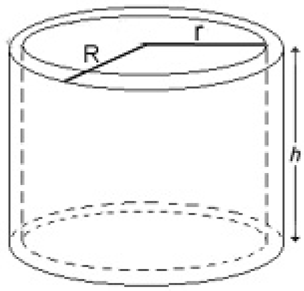
We know the area of the bases from part b. The lateral face is a rectangle whose height is the same as the height of the cylinder and its width is the perimeter of the base. (Imagine making a cylinder out of paper. When you unroll it, you'll have a rectangle with these dimensions). As scratch work, we would compute

$$\begin{aligned}\text{area of lateral face} &= (\text{height of cylinder}) \cdot (\text{perimeter of base})\\ &= (6 \text{ inches}) \cdot (96 + 6\pi \text{ inches})\\ &= 576 + 36\pi \text{ square inches.}\end{aligned}$$

Returning to our formula,

$$\begin{aligned}\text{surface area} &= 2 \cdot (\text{bases}) + 1 \cdot (\text{lateral face})\\ &= 2 \cdot (864 + 9\pi \text{ square inches}) + 1 \cdot (576 + 36\pi \text{ square inches})\\ &= (1728 + 18\pi \text{ square inches}) + (576 + 36\pi \text{ square inches})\\ &= 2304 + 54\pi \text{ square inches.}\end{aligned}$$

3. Suppose you have a cylinder of height h and radius R and you bore out a centrally-located cylinder of radius r. The result is called a cylindrical shell.



3a. Write an equation for the volume of the cylindrical shell in the picture.

Solution: The volume of the shape if nothing was removed would be $\pi R^2 h$ (area of the circular base times the height of the cylinder). The volume of the removed cylinder is $\pi r^2 h$. So, the remaining area is $\pi R^2 h - \pi r^2 h$, which you might rewrite as $\pi h(R^2 - r^2)$.

3b. Write an equation for the surface area of the cylindrical shell in the picture.

Solution: The surface area is made up of two bases, an outside lateral face, and an inside lateral face. So,

$$\text{surface area} = 2\cdot(\text{area of base}) + 1\cdot(\text{area of outside lateral face}) + 1\cdot(\text{area of inside lateral face}).$$

If the base had nothing removed, the area would be πR^2. The area of the removed portion is πr^2. So, the remaining area is $\pi R^2 - \pi r^2$.

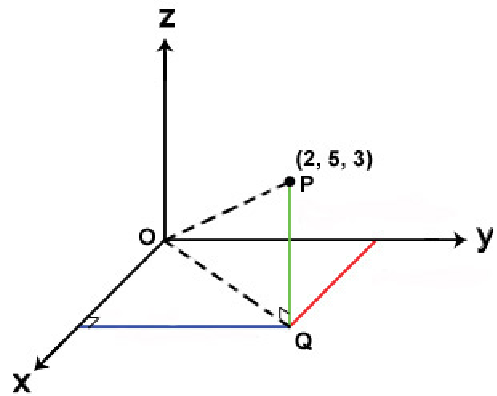
The outside lateral face is a rectangle with height h and width $2\pi R$ (the circumference of the larger circle). So, the total area is $2\pi Rh$.

The inside lateral face is a rectangle with height h and width $2\pi r$ (the circumference of the smaller circle). So, the total area is $2\pi rh$.

Putting all the pieces together,

$$\begin{aligned} \text{surface area} &= 2\cdot(\text{area of base}) + 1\cdot(\text{area of outside lateral face}) + 1\cdot(\text{area of inside lateral face}) \\ &= 2\cdot(\pi R^2 - \pi r^2) + 1\cdot(2\pi Rh) + 1\cdot(2\pi rh) \\ &= 2\pi R^2 - 2\pi r^2 + 2\pi Rh + 2\pi rh. \end{aligned}$$

4. Repeated application of the Pythagorean Theorem can be used to find the distance between points in higher dimensions.



4a. What is the length of the line segment $\overline{OQ}$?

Solution: The triangle on the floor is a right triangle whose legs are length 2 and 5 (the x-coordinate and the y-coordinate of P). The hypotenuse is the line segment $\overline{OQ}$, which is what we are interested in finding. If we let c be the length of $\overline{OQ}$, then the Pythagorean Theorem tells us

$$\begin{aligned} 2^2 + 5^2 &= c^2 \\ 4 + 25 &= c^2 \\ 29 &= c^2 \\ \sqrt{29} &= c. \end{aligned}$$

4b. What is the length of the line segment $\overline{OP}$? (This is the distance between the origin and the point P.)

Solution: The triangle perpendicular to the floor (the one whose vertices are O, Q, and P) is a right triangle whose legs are length $\sqrt{29}$ (this is the leg on the floor) and 3 (this is the z-coordinate of P). The hypotenuse is the line segment $\overline{OP}$, which is what we are interested in finding. If we let c be the length of $\overline{OP}$, then the Pythagorean Theorem tells us

$$\begin{aligned} (\sqrt{29})^2 + 3^2 &= c^2 \\ 29 + 9 &= c^2 \\ 38 &= c^2 \\ \sqrt{38} &= c. \end{aligned}$$

4c. Use this idea to find the distance between the origin and the point (2,5,3,7) in four-dimensional space.

Solution: We have limited power to visualize four-dimensional space, but algebra lets us explore it, nonetheless. The triangle perpendicular to all of three-dimensional space (whatever that means) is a right triangle whose legs are length $\sqrt{38}$ (this is the leg contained in the first three dimensions) and 7 (this is the leg poking out into the fourth dimension). The hypotenuse is the distance we are interested in finding. If we let c be this distance, then the Pythagorean Theorem tells us

$$\begin{aligned}(\sqrt{38})^2 + 7^2 &= c^2\\ 38 + 49 &= c^2\\ 87 &= c^2\\ \sqrt{87} &= c.\end{aligned}$$

The beauty of mathematics is that it lets you explore what you can never personally experience. You can take that as a sign of impracticality, or you can take it as the beginnings of understanding a reality much richer than the one we experience day-to-day.

5. A circle has the points (-1, 2) and (4, -3) as opposite enpoints of one of its diameters.

5a. What is the radius of the circle?

Solution: The distance between the given points will give us the diameter of the circle. Using the distance formula, the diameter has length

$$\begin{aligned}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= \sqrt{(4 - (-1))^2 + (-3 - 2)^2}\\ &= \sqrt{5^2 + (-5)^2}\\ &= \sqrt{25 + 25}\\ &= \sqrt{50}.\end{aligned}$$

Since the radius is just half the diameter, the radius has length $\frac{\sqrt{50}}{2}$.

5b. What is the center of the circle?

Solution: The center of the circle is the midpoint of the two endpoints of the diameter (the end points of any diameter would have worked). The midpoint formula tells us the center is at

$$\begin{aligned}\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) &= \left(\frac{-1+4}{2},\frac{2+(-3)}{2}\right)\\ &= \left(\frac{3}{2},-\frac{1}{2}\right).\end{aligned}$$

5c. What is the equation of the circle?

Solution: Now that we have the radius and the center, we just put all this information in the generic equation of a circle.

$$\begin{aligned}(x-h)^2+(y-k)^2 &= r^2\\ (x-\frac{3}{2})^2+(y-(-\frac{1}{2}))^2 &= \left(\frac{\sqrt{50}}{2}\right)^2\\ (x-\frac{3}{2})^2+(y+\frac{1}{2})^2 &= \frac{50}{4}\\ (x-\frac{3}{2})^2+(y+\frac{1}{2})^2 &= \frac{25}{2}.\end{aligned}$$

5d. Use your equation in part c to determine if the point (1, -1) lies on the circle.

Solution: The equation in part c describes *all* points (x,y) that lie on the circle. If we plug in values for x and y and get a false statement, then they do not lie on the circle. If we get a true statement, then they do.

$$\begin{aligned}(x-\frac{3}{2})^2+(y+\frac{1}{2})^2 &\stackrel{?}{=} \frac{25}{2}\\ (1-\frac{3}{2})^2+(-1+\frac{1}{2})^2 &\stackrel{?}{=} \frac{25}{2}\\ (-\frac{1}{2})^2+(-\frac{1}{2})^2 &\stackrel{?}{=} \frac{25}{2}\\ \frac{1}{4}+\frac{1}{4} &\stackrel{?}{=} \frac{25}{2}\\ \frac{1}{2} &\stackrel{?}{=} \frac{25}{2}.\end{aligned}$$

The last line is clearly false, so we conclude that the point $(1,-1)$ does not lie on the circle.