University of South Carolina
Math 222: Math for Elementary Educators II
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Section 002
Fall 2010

Quiz 1 Solutions

1. Give a short answer to each of the following and explain your reasoning. Use pictures if it helps to convey your thoughts.

a. Can skew lines have a point in common? Can skew lines be parallel?

Solution: Suppose for a moment that two lines have a point A in common (we will find out this can't actually happen). Pick a point B on the first line and a point C on the second line (both different from A). We can define the plane ABC that contains both the lines $\overline{AB}$ and $\overline{BC}$. This is impossible, though, since the lines were supposed to be skew (i.e. *no* plane contains both the lines). Since believing that two skew lines have a point in common leads to an impossible situation, we conclude that two skew lines cannot have a point in common. (This technique of rejecting a belief because it leads to an impossibility is called proof by contradiction or *reductio ad absurdum* if you're a stuck-up philosopher.)

Parallel lines, by definition, must lie in the same plane. Skew lines, by definition, cannot lie in the same plane. Thus, skew lines cannot be parallel.

b. Is it possible to locate four points in a plane such that the number of lines determined by the points is not exactly 1, 4, or 6?

Solution: If all four points are collinear, then any two points you choose to define your line will result in the same line. If three points (say, A, B, and C) are collinear and one is not (call it D), then four lines different lines can be formed ($\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ are all distinct, while $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ are all the same). If no three points are collinear, then any choice of two points gives a different line, giving six possible lines ($\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$ are all different lines in this situation). Having exhausted all possible arrangements of the points, we conclude that 1, 4, and 6 are the only possibilities for the number of lines we can form. (This answer will make much more sesne if you draw a picture to accompany each of the three possible configurations of the points.)

c. Is it possible to locate four points in three-dimensional space such that the number of planes determined by the points is not exactly 1 or 4?

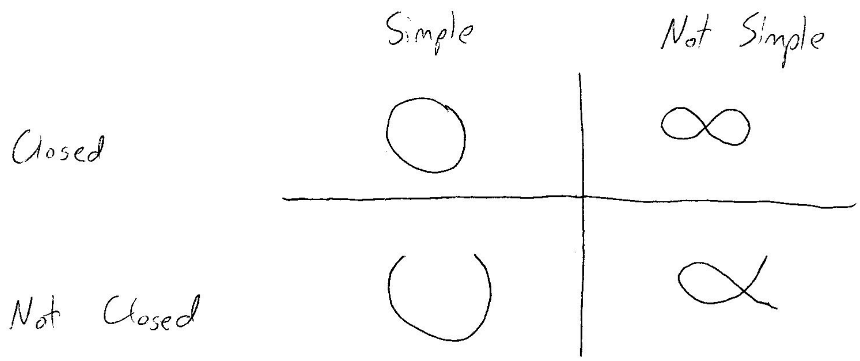
Solution: If all four points are coplanar, then any three points you choose to define your plane will result in the same plane. If three points (say, A, B, and C) are coplanar and one is not (call it D), any choice of three points gives a different plane, giving four possible planes ($\overline{ABC}$, $\overline{ABD}$, $\overline{ACD}$, and $\overline{BCD}$ are all different planes in this situation). Having exhausted all possible arrangements of the points, we conclude that 1 and 4 are the only possibilities for the number of planes we can form. (This answer will make much more sesne if you draw a picture to accompany each of the two possible configurations of the points.)

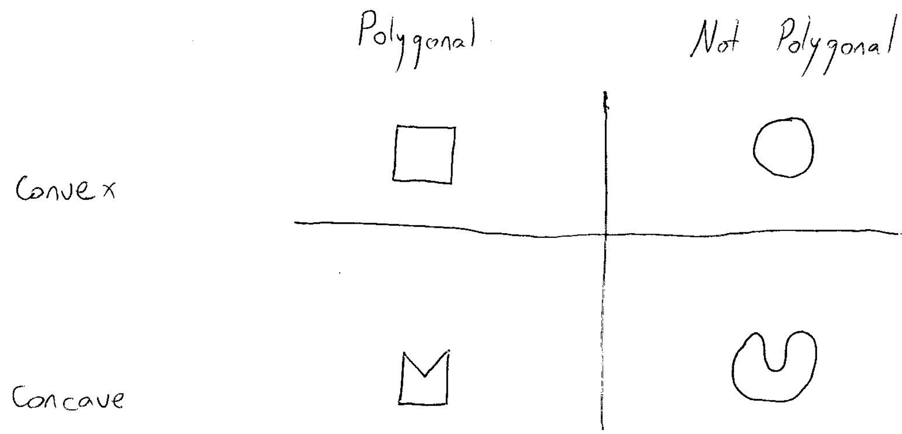
2. Draw a picture for each of the possibilities described below. Draw from multiple perspectives or elaborate with words it helps clarify your three-dimensional drawings.

a. All four possibilities for curves that are simple/not simple, closed/not closed

b. All four possibilities for curves that are convex/concave, polygonal/not polygonal

Solution:

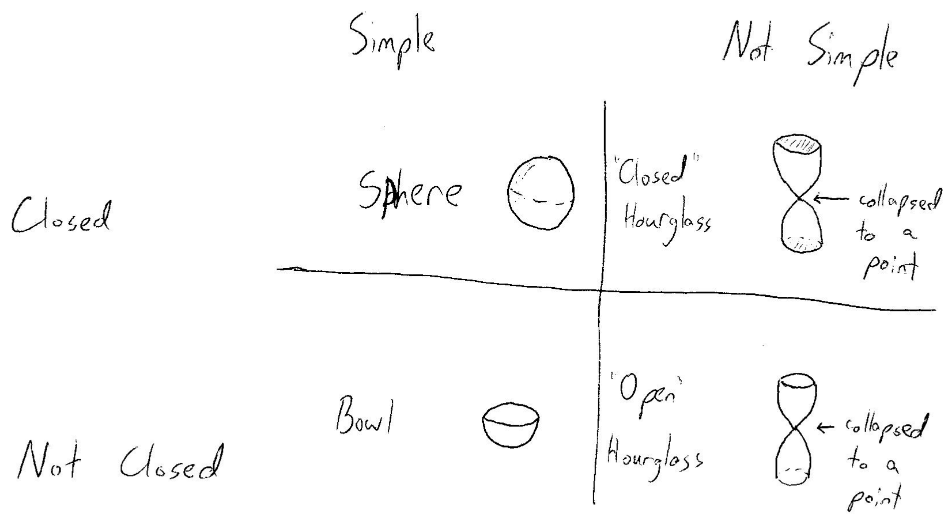

Simple
Not Simple
Closed
Not Closed

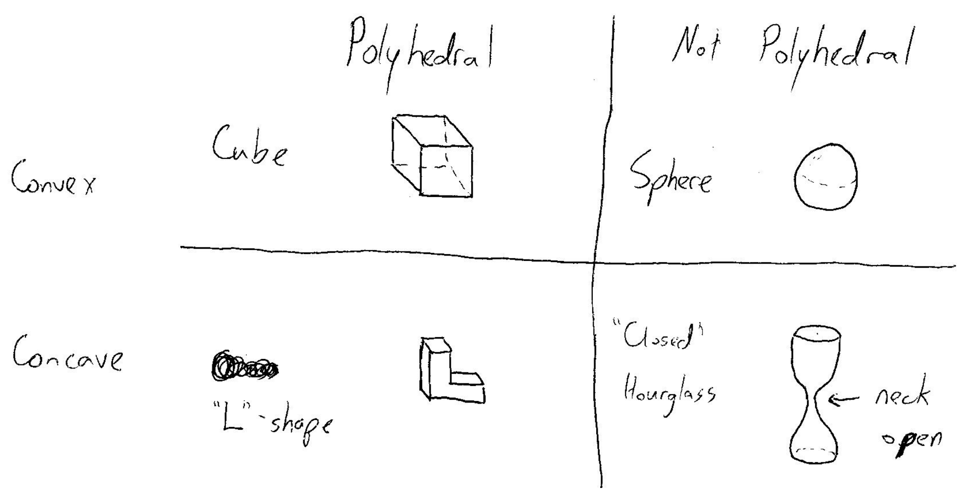

Polygonal
Not Polygonal
Convex
Concave

c. All four possibilities for surfaces that are simple/not simple, closed/not closed

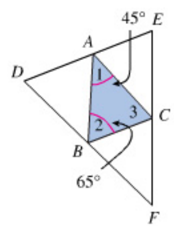
d. All four possibilities for surfaces that are convex/concave, polyhedral/not polyhedral

Solution:





3. In the following picture, $\overleftrightarrow{DE} \| \overleftrightarrow{BC}$, $\overleftrightarrow{EF} \| \overleftrightarrow{AB}$, and $\overleftrightarrow{DF} \| \overleftrightarrow{AC}$. Find the measure of all missing angles. Show your work.



Solution: Since there are 180 degrees in every triangle, angle $\angle 3$ has measure

$$180° - 65° - 45° = 70°.$$

There are many parallel lines that we can use to work in a transversal setting. For example, consider the lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$. Since these are parallel, we know that angle $\angle 2$ is congruent to angle $\angle BAD$ and angle $\angle 3$ is congruent to angle $\angle CAE$ (they are both pairs of alternate interior angles). Using this technique two more times (with the other two pairs of parallel sides), we get six of the nine missing angles. At this point, every triangle will have two of its three angles filled in, and you can again use the fact that every triangle has 180 degrees to find the final missing angle in each triangle.

$$m(\angle 1) = m(\angle ABD) = m(\angle ACE) = m(\angle BFC) = 45°$$

$$m(\angle 2) = m(\angle BAD) = m(\angle AEC) = m(\angle BCF) = 65°$$

$$m(\angle 3) = m(\angle ADB) = m(\angle CAE) = m(\angle CBF) = 70°$$

4. Using only the basic definitions of each shape, decide whether the following statements are true or false. If a statement is true, explain why it is true. If it is false, give a counterexample (with explanation) showing it is false.

a. Every obtuse triangle is scalene.

Solution: False. One can easily draw an iscosceles triangle with an obtuse angle. This triangle will be obtuse, but not scalene.

b. Every parallelogram is a trapezoid.

Solution: True. If you have a parallelogram in your hand, it will have two pairs of parallel sides. If a trapezoid inspector comes by, s/he will only check to make sure it has at least one pair of parallel sides, so your parallelogram will pass the test.

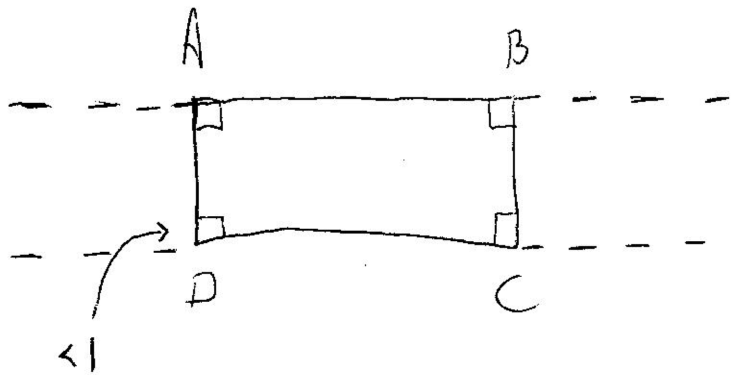
c. No rhombus is a kite.
Solution: False. Just like in part b, imagine you have a rhombus in your hand. All of its sides will be the same length. If a kite inspector comes in, s/he will verify that it has two pairs of sides of equal length, which it does. So, in fact, *every* rhombus is a kite, making the statement super false.

d. If a quadrilateral is both a rectangle and a rhombus, then it is a square.
Solution: True. Think of the things the square inspector is going to check:

1. Does the quadrilateral have four right angles?
2. Are opposite sides parallel? (This is actually a bit redundant, since having four right angles automatically makes opposite sides parallel.)
3. Does the quadrilateral have sides all the same length?

Since your quadrilateral is a rectangle, it will pass the first two tests. Since your quadrilateral is also a rhombus, it will pass the third test, and so it will receive its square license from the inspector.

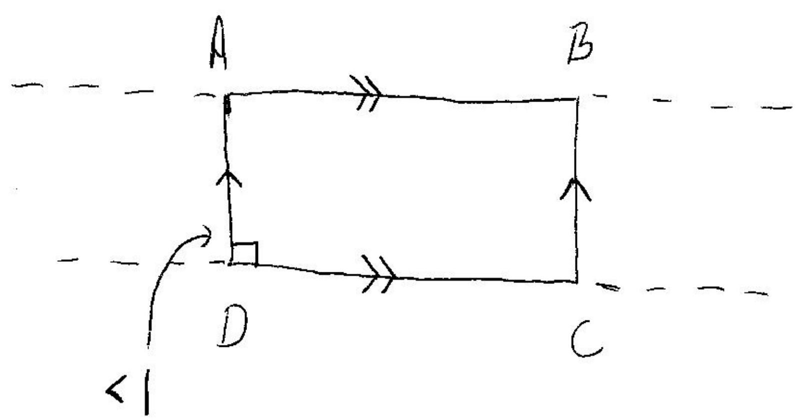
5. A *failbox* is a quadrilateral whose interior angles are all congruent. The failbox fails because it is actually just another way to define rectangle.
a. Show that every failbox is a rectangle.
Solution: Suppose you have a failbox in your hand. Since all the interior angles are congruent and there are $(4-2)\cdot 180 = 360$ degrees in every quadrilateral, each angle must be $\frac{360}{4} = 90$ degrees. So, we have this picture so far.



The only thing left to show is that opposite sides are parallel. We can do this by establishing any one of the facts like "In a transversal, ____ angles are congruent." Now, the angle I marked $\angle 1$ is 90 degrees, since it forms a straight line together with angle $\angle ADC$. Since angle $\angle BAD$ is also 90 degrees, we have alternate interior angles being equal. This means that the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ really are parallel. A very similar argument will tell us that $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are also parallel. Taking all this together, we see that the quadrilateral $ABCD$ really is a rectangle.

b. Show that every rectangle is a failbox.

Solution: Under the minimal definitions, we have the following picture of a rectangle.



Our goal is to show the the other three interior angles are also right angles. We do this in almost the same way as in part a. Since angle $\angle 1$ together with angle $\angle ADC$ make a straight line, we know that angle $\angle 1$ is a right angle. Since $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, we know angle $\angle BAD$ is also a right angle (since opposite interior angles of a transversal of parallel lines are equal). A similar argument can show that the other two interior angles of the quadrilateral are also right angles. Taking all this together, we see that the quadrilateral $ABCD$ really is a failbox.

Take a moment to notice the difference between the solution of parts a and b. In part a, we knew that the alternate interior angles happened to be equal, but not that the lines were parallel. However, alternate interior angles being equal is one way to determine that lines are parallel, so we got the conclusion we wanted. In part b, we knew the lines happened to be parallel, but we didn't know about the alternate interior angles. However, we remember that whenever the lines are parallel, alternate interior angles will be equal, so we got the conclusion we wanted.

6. A *uniform polyhedron* is any polyhedron whose faces are regular polygons (not necessarily all of the same type) and have the same number of each type of face meeting at every vertex. For example, a soccer ball is a uniform polyhedron, since all the faces are either regular pentagons or regular hexagons and there is always one pentagon and two hexagons meeting at any vertex.

a. When is a prism a uniform polyhedron? (Note: Don't forget to consider right and oblique.)

Solution: A prism is made out of two polygonal bases and parallelogram lateral faces. Since all the faces of a uniform polyhedron have to be regular, we know the bases have to be regular polygons and the lateral faces have to be regular parallelograms (i.e. squares). The condition about having the same number of each type of face meeting at a every vertex is already satisfied by any prism, so we're done. (Note: Since the lateral faces are squares, the prism is forced to be a right prism.)

b. When is a pyramid a uniform polyhedron? (Note: Don't forget to consider right and oblique.)

Solution: For the same reasons as in part a, the base and lateral faces must be regular polygons. In a pyramid, however, the lateral faces are triangles. Think about how the apex differs from the other vertices. No matter what base you pick, the apex will see only triangles (it's the place where all the lateral faces meet). The other vertices will see some triangles as well as the base. Since every vertex is supposed to see exactly the same number and type of faces, we're forced to use a triangle as the base. (Note: Since the lateral faces are equilateral triangles, the pyramid is forced to be a right prism.)

c. How many uniform polyhedra could there be using only triangles and pentagons?

A regular triangle has $(3-2)\cdot 180 = 180$ degrees total, which means each angle measures $\frac{180}{3} = 60$ degrees. A regular pentagon has $(5-2)\cdot 180 = 540$ degrees total, which means each angle measures $\frac{540}{5} = 108$ degrees. There are two things to keep in mind when forming polyhedra of any kind:

1. There must be at least three faces meeting at a vertex (you cannot close your surface with only one or two faces at some vertex).

2. The sum of the interior angles meeting at a vertex must be less than 360 degrees. Otherwise, that vertex is completely flat (meaning it isn't a vertex at all).

With just these two guidelines, we narrow down the infinite number of possibilities to something much more manageable.

Faces at Vertex	**Sum of Angles at Vertex**	**Verdict**
3 Triangle, 0 Pentagon	$3 \cdot 60° = 180°$	Possible
4 Triangle, 0 Pentagon	$4 \cdot 60° = 240°$	Possible
5 Triangle, 0 Pentagon	$5 \cdot 60° = 300°$	Possible
6 Triangle, 0 Pentagon	$6 \cdot 60° = 360°$	Impossible
2 Triangle, 1 Pentagon	$2 \cdot 60° + 1 \cdot 108° = 228°$	Possible
3 Triangle, 1 Pentagon	$3 \cdot 60° + 1 \cdot 108° = 288°$	Possible
4 Triangle, 1 Pentagon	$4 \cdot 60° + 1 \cdot 108° = 348°$	Possible
5 Triangle, 1 Pentagon	$5 \cdot 60° + 1 \cdot 108° = 408°$	Impossible
1 Triangle, 2 Pentagon	$1 \cdot 60° + 2 \cdot 108° = 276°$	Possible
2 Triangle, 2 Pentagon	$2 \cdot 60° + 2 \cdot 108° = 336°$	Possible
3 Triangle, 2 Pentagon	$3 \cdot 60° + 2 \cdot 108° = 396°$	Impossible
0 Triangle, 3 Pentagon	$3 \cdot 108° = 324°$	Possible
1 Triangle, 3 Pentagon	$1 \cdot 60° + 3 \cdot 108° = 384°$	Impossible
0 Triangle, 4 Pentagon	$4 \cdot 108° = 432°$	Impossible

Larger number of faces obviously won't work, so we can stop. Our list shows nine possibilities for uniform polyhedra.