

University of South Carolina
Math 222: Math for Elementary Educators II
Instructor: Austin Mohr
Section 002
Fall 2010

Midterm Exam Solutions

Please write your solutions (including work) on the blank paper, not the test sheet. For problems involving trigonometry, make sure your calculator is in degree mode, not radian mode.

Definitions

Define each of the following terms.

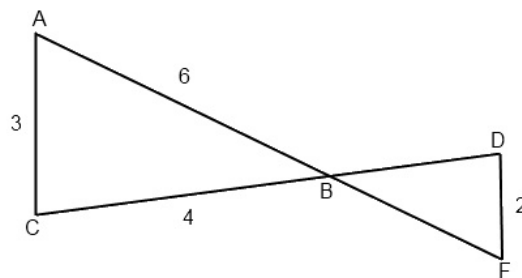
1. similar triangles
2. simple (three-dimensional version)
3. convex
4. polygon
5. polyhedron
6. prism

I would like you to digest the definitions from the notes or the text, so I will not include them here.

Computation

Provide the requested information for each question. Show all your work.

7. Refer to the figure below for the following questions. You may assume that side \overline{AC} is parallel to side \overline{DF} .

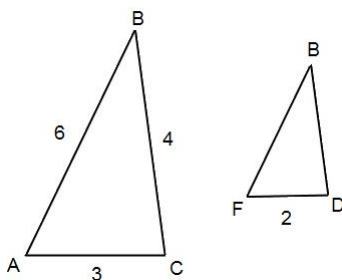


- a. Show that the big triangle on the left is similar to the small triangle on the right.

Solution: We need to establish one of the similarity rules (AA, SAS, or SSS). We only have one pair of sides in common, so SAS and SSS are impossible to establish. That leaves us with AA. Angle $\angle ABC$ and angle $\angle DBF$ are vertical angles, so they are congruent. Since \overline{AC} and \overline{DF} are parallel, we can use transversals to find more angles in common. In particular, angle $\angle BAC$ and angle $\angle BFD$ are alternate interior angles, so they are congruent. Since we have two pairs of angles in common, we know the two triangles are similar by the AA similarity rule.

- b. Sketch the two triangles side-by-side (with the vertices labelled) so that they are oriented the same way.

Solution: We just need to make sure that congruent angles get put into the same position. From what we say in the last problem, that means the vertices B and B should be in the same position, A and F should be in the same position, and C and D should be in the same position.



- c. Find the length of side \overline{BF} .

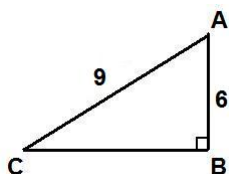
Solution: Since the triangles are similar, we know that their sides are proportional. That means the fraction

$$\frac{\text{Side in Big Triangle}}{\text{Related Side in Small Triangle}}$$

will be the same number no matter which pair of sides you use. That means we can do the following:

$$\begin{aligned} \frac{3}{2} &= \frac{6}{BF} \\ 3BF &= 12 && \text{(cross-multiplying)} \\ BF &= 4. \end{aligned}$$

8. Find all missing sides and angles in the triangle below. You may *not* use the Pythagorean Theorem.



Solution: To find angle $\angle A$, we need to relate it to the \overline{AB} (which is adjacent to angle $\angle A$) and \overline{AC} (which is the hypotenuse) using the appropriate trigonometric function. Since adjacent and hypotenuse are involved, we need to use cosine.

$$\cos(\angle A) = \frac{6}{9}.$$

To get angle $\angle A$ by itself, we take the inverse cosine of both sides of the equation.

$$\begin{aligned}\cos^{-1}(\cos(\angle A)) &= \cos^{-1}\left(\frac{6}{9}\right) \\ \angle A &= 48.2^\circ.\end{aligned}$$

To find angle $\angle C$, we need to relate it to the \overline{AB} (which is opposite of angle $\angle C$) and \overline{AC} (which is the hypotenuse) using the appropriate trigonometric function. Since opposite and hypotenuse are involved, we need to use sine.

$$\sin(\angle C) = \frac{6}{9}.$$

To get angle $\angle C$ by itself, we take the inverse sine of both sides of the equation.

$$\begin{aligned}\sin^{-1}(\sin(\angle C)) &= \sin^{-1}\left(\frac{6}{9}\right) \\ \angle C &= 41.8^\circ.\end{aligned}$$

To find side \overline{BC} , we can choose any angle and any other side and relate them all with the appropriate trigonometric function. For example, I might pick angle $\angle C$ and side \overline{AB} . Relative to angle $\angle C$, side \overline{BC} is adjacent and side \overline{AB} is opposite. Since adjacent and opposite are involved, we need to use tangent. Let's call the length of side \overline{BC} x . We can set up the equation

$$\begin{aligned}\tan(41.8^\circ) &= \frac{6}{x} \\ .89 &= \frac{6}{x} \\ .89x &= 6 && \text{(multiply both sides by } x \text{ to clear denominator)} \\ x &= 6.7.\end{aligned}$$

9. The following two questions are unrelated. You will not use part a to solve part b.

- a. A factory is trying to decide whether to produce cogs or sprockets. The profit for producing cogs is modelled by the equation $y = x - 500$, where x is the number of cogs produced and y is the profit in dollars. Similarly, the profit for producing sprockets is modelled by the equation $y = 3x - 1000$. What is the level of production (i.e. how many units are being produced) if the profit is the same for both cogs and sprockets? How much profit is being made at this level of production?

Solution: We want to find out where the two equations are equal. That is, we want to find the single point where the lines intersect. To do that, we first set the two equations equal to each other and solve for x .

$$x - 500 = 3x - 1000$$

$$x + 500 = 3x$$

$$500 = 2x$$

$$250 = x.$$

This means the x -coordinate of the intersection is 250. To find the y -coordinate that goes with this x , we plug 250 into either equation (we should get the same answer for either equation, since the y -coordinates are supposed to be the same here).

$$y = x - 500$$

$$y = 250 - 500$$

$$y = -250.$$

This means the y -coordinate of the intersection is -250 . In the context of the problem, this means that you produce exactly 250 things (cogs or sprockets), you lose 250 dollars either way.

- b. Water is being added to a tank at a constant rate. Four seconds after the filling began, the tank contained 20 gallons of water. Ten seconds after filling began, the tank contained 41 gallons. Write a linear equation to describe the change in the volume of water over time. Be sure to specify the meaning of the variables x and y .

Solution: Let x be the time and y be the volume of water. Under this interpretation, the problem is asking you to find the equation of the line through $(4, 20)$ and $(10, 41)$. Every line has an equation of the form $y = mx + b$, so we just need to find m and b .

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{41 - 20}{10 - 4} \\ &= \frac{21}{6} \\ &= \frac{7}{2}. \end{aligned}$$

So, our equation looks like $y = \frac{7}{2}x + b$. Next, we can solve for b by plugging in either point.

$$\begin{aligned}y &= \frac{7}{2}x + b \\20 &= \frac{7}{2} \cdot 4 + b \\20 &= 14 + b \\6 &= b.\end{aligned}$$

So, our final equation is $y = \frac{7}{2}x + 6$.

Short Answer

Give a short response to each of the following questions. Do *not* give a full proof.

10. How many planes are determined by four points that are not all coplanar? Explain.

Solution: We have four points A , B , C , and D that are not all coplanar. That means that *every* choice of three of them makes a unique plane (there are no redundancies). So, we list out all the possibilities: ABC , ABD , ACD , and BCD . Remember, we only care about which points are chosen, not the order in which they are chosen. This is why, for example, we list ABC but not BCA , CBA , etc. Our list shows that there are four different planes that we can define.

11. Is every trapezoid a rhombus? Is every rhombus a trapezoid? Explain.

Solution: A trapezoid must have at least one pair of parallel sides, while a rhombus must have two pairs of parallel sides (and all sides must be equal, but that part isn't important to us).

Suppose you have a trapezoid. Don't try to visualize it; it will only mislead you. Rely only on the definition of trapezoid. You know that the shape (that you aren't visualizing) has at least one pair of parallel sides. Does it have to have two pairs of parallel lines? Of course not, you can draw many examples (now you are allowed to visualize an example).



Is this a trapezoid? Yes, it has at least one pair of parallel sides. Is it a rhombus? No, it does not have two pairs of parallel sides. Because of this

example, you know that not every trapezoid is a rhombus. (On a related note, many of you wrote “Every trapezoid is not a rhombus” when you meant “Not every trapezoid is a rhombus”. These are *extremely* different sentences. To put it in more familiar terms, consider the sentences “Every dog is not a poodle” and “Not every dog is a poodle”. The first says that every dog in the world is not a poodle, which is not true (some dogs really are poodles). The second says that there are some dogs that aren’t poodles, which is true (labradors are dogs that aren’t poodles).)

Suppose now that you have a rhombus. Again, resist the urge to visualize it. Just think about the definition. You know the shape must have two parallel sides. Does it have to have at least one parallel side? Of course it does. If it already has two, then it definitely has at least one. By this reasoning, *every* rhombus is a trapezoid.

- 12.** Can there be a regular polyhedron with three hexagons meeting at every vertex? Why or why not?

Solution: The criterion we need to check is that the sum of the degrees at each vertex of the polyhedron is less than 360° . Since there are three regular hexagons meeting at every vertex, the total number of degrees of angles at the vertex is equal to three times whatever the measure of a *single* angle in a regular hexagon is. Let’s go on a tangent and find the measure of an angle in a regular hexagon.

We have a formula for the *total* number of degrees in an n -gon: $(n - 2) \cdot 180^\circ$. Since we’re talking about a hexagon, n is 6. This means the *total* number of degrees in the hexagon is $(6 - 2) \cdot 180^\circ = 720^\circ$. Using this, how can we find the measure of a *single* interior angle? Since the hexagon is regular, that means all the angles are the same. So, we can just take the total 720° and divide by 6, which gives 120° for a *single* interior angle.

Now, let’s return to the original question. How many degrees are meeting at the vertex of a polyhedron if there are three angles of a hexagon meeting there? It must be $3 \cdot 120^\circ = 360^\circ$, which is too many degrees (with this many degrees, the vertex won’t be a vertex at all; it will be flat). So, there cannot be a regular polyhedron with three hexagons meeting at every vertex.

- 13.** Explain why certain triangle congruence rules are “good” (i.e. they can be used to tell two triangles are congruent), while other congruence rules are “bad” (i.e. they give no information about the congruence of two triangles).

Solution: A congruence rule (for triangles or for any shape) is good if, given two shapes satisfying the rule, all other aspects of the shape are forced. For example, with SSS congruence in triangles, knowing that two triangles have all pairs of associated sides equal means that all pairs of associated angles will be equal, as well.

A congruence rule is bad if, given two shapes satisfying the rule, it is possible that some other aspect differs between the two shapes. For example, AAA is a bad congruence rule in triangles because there are many (in fact, infinitely-

many) 30-60-90 triangles. Knowing that two triangles are 30-60-90 (and so satisfy AAA) tells you basically nothing about the length of the sides.