University of South Carolina
Math 221: Math for Elementary Educators
Instructor: Austin Mohr
Section 001
Spring 2010

Test 1

Help Prime Minister Mittens govern Talking Cat Island by analyzing the following three word problems. For each problem

a. tell what operation is involved ($+$, $-$, $\times$, or $\div$),
b. give the name of an appropriate model for the problem, and
c. draw a picture demonstrating the solution.

1. Talking Cat Island is home to 24 thousand cats. If a district contains 3 thousand cats, how many districts are required? (Hint: Work with the numbers 24 and 3, not 24,000 and 3,000.)

Solution: We are given the total number of objects and the desired size of each group. We want to know how many groups are required. This indicates the repeated subtraction model of division.

2. Every second, 6 saucers of milk are consumed by the citizens of the island. How many saucers do they drink in 5 seconds?

Solution: The problem above involves multiplication. One could make a case for either repeated addition (each second, we get another group of 6) or the array model (each row contains 6 saucers, and we have a row representing each second). Personally, I would argue that the repeated addition model lends itself more naturally to the linear nature of time (6 saucers now, then 6 more, then 6 more, . . .), whereas the array structure seems arbitrarily imposed.

3. It takes one crew of catstruction workers 14 hours to build a scratching post, while it takes another crew only 11 hours to build the same post. How much longer does the first crew require than the second?

Solution: Whenever time is the thing we are actually interested in measuring (as opposed to the previous problem, where it simply helped us define our groups), the number line model tends to work best. Specifically, the number line model of subtraction is appropriate here.

[2] 4. Describe the following sets of numbers.

a. Natural Numbers

Solution: $\{1, 2, 3, \ldots\}$

b. Whole Numbers

Solution: $\{0, 1, 2, 3, \ldots\}$

c. Integers

Solution: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

d. Rational Numbers

Solution: $\{\frac{a}{b} \mid a, b \text{ integers}, b \neq 0\}$

Loosely speaking, this says the rationals are all fractions you can make with numerator and denominator from the integers with the exception that the denominator is not allowed to be 0. Of course, many of these fractions will be equivalent (for example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$, etc.).

Another equally valid description would be to say that the rationals are those numbers whose decimal representations either terminate (such as 1.5) or repeat (such as $2.\overline{3}$). Even this description has some redundancy, however (remember $.\overline{9} = 1$).

e. Real Numbers

Solution: Any number that can be written as a decimal (including non-terminating, non-repeating ones) is a real number. What kinds of numbers aren't real numbers? The next step up is the complex numbers, which incorporates things like $\sqrt{-1}$ that don't make any sense in the reals.

5a. Convert 342_5 to base 10.

Solution: The number 342_5 indicates 3 flats, 4 sticks, and 2 singles of the base 5 variety. Each flat gives 25 singles and each stick gives 5 singles. So, we have a total of $3 \times 25 + 4 \times 5 + 2 = 97$ singles. In other words, $342_5 = 97_{10}$.

5b. Convert 113_{10} to base 5.

Solution: The number 113_{10} can be viewed as 113 singles (here, I use 113 to really mean the "one hundred thirteen" of base 10). We can regroup this into base 5 blocks using 4 flats (which uses up 100 singles), 2 sticks (which uses 10 more singles), and leaving the remaining 3 singles alone. In other words, $113_{10} = 423_5$.

[2] 6. Evaluate each of the following expressions using the specified method.

a. $232_5 + 244_5$ using the partial sums method

Solution:

$$\begin{array}{rrrrr} & & 2 & 3 & 2_5 \\ + & & 2 & 4 & 4_5 \\ \hline & & 4 & 0 & 0_5 \\ & & 1 & 2 & 0_5 \\ + & & & 1 & 1_5 \\ \hline & 1 & 0 & 3 & 1_5 \end{array}$$

b. $231_5 - 133_5$ using the equal addends method

Solution: Adding 12_5 to both numbers changes the problem to $243_5 - 200_5 = 43_5$. If you didn't see the 12_5 immediately, you could have done it instead in two steps: first 2_5 and then 10_5.

c. $21_5 \times 23_5$ using the partial products method

Solution: The multiplication can be performed in any order. I will do it the same way as in the standard algorithm.

$$\begin{array}{rrrr} & & 2 & 1_5 \\ \times & & 2 & 3_5 \\ \hline & & & 3_5 \\ & & 1 & 1_5 \\ & & 2 & 0_5 \\ + & 4 & 0 & 0_5 \\ \hline & 4 & 3 & 4_5 \end{array}$$

d. $421_5 \div 3_5$ using the standard algorithm

Solution: For the problem below, all numbers are assumed to be in base 5. (Please forgive the poor typesetting.)

$$
\begin{array}{cc|ccc}
 & & 1 & 2 & 2 \\
\hline
3 & | & 4 & 2 & 1 \\
 & - & 3 & & \\
\hline
 & & 1 & 2 & \\
 & - & 1 & 1 & \\
\hline
 & & & 1 & 1 \\
 & - & & 1 & 1 \\
\hline
 & & & & 0
\end{array}
$$

7. Short Answer

a. Consider the set of even whole numbers $\{0, 2, 4, 6, \cdots\}$. Is this set closed under addition? Is it closed under subtraction? Explain.

Solution: An even whole number plus an even whole number is again an even whole number (for example, $4 + 8 = 12$), so the set of even whole numbers is closed under addition. The set is not closed under subtraction, however, since, for example, $0 - 2$ is a subtraction problem involving two even whole numbers, but the solution of -2 is not an even whole number (no negative number belongs to our set).

We can be more precise and actually prove our claim that the set of even whole numbers is closed under addition. Remember that when we say a number is even, we really mean that it has a factor of 2 in it. So, we can represent our two arbitrary even whole numbers as $2 \cdot a$ and $2 \cdot b$, where a and b are regular whole numbers. (As an example, 6 is even because it is really $2 \cdot 3$, 100 is even because it is really $2 \cdot 50$, and so on.) Now, what happens when we add these two numbers together?

$$2a + 2b = 2(a + b) \qquad \text{(by distributivity)}$$

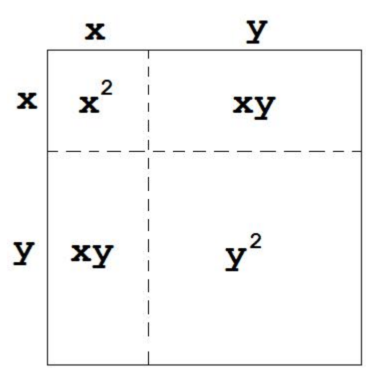
So, if we add two even whole numbers, we get another even whole number, since the sum is just 2 times the whole number $a + b$. Of course, even this claim relies on the fact that the whole numbers are closed under addition (we want $a + b$ to actually be a whole number, after all).

b. Why do we say that, for example, $0 \div 1 = 0$ (and so is defined) but $1 \div 0$ is undefined?

Solution: Let's consider these problems in the context of the repeated subtraction model. In $0 \div 1$, we have 0 objects and we want to know how many groups of size 1 we can make. Obviously, we can't make any groups (we don't even have any objects to put into groups), so our answer should be 0. On the other hand, $1 \div 0$ means we have one object and we want to know how many groups of size 0 we can make. Somehow, this question doesn't even make sense. What is a group of size 0? What would limit us from making as many as we want? No matter how many we make, we'll always have that 1 object remaining (since we never actually put it in a group). No matter how hard we try, we can't find a numerical answer that makes any sense, so we just say that the answer is undefined.

c. Prove with a picture that $(x+y)^2 = x^2 + 2xy + y^2$. (Hint: Remember that $(x+y)^2$ is the same as $(x+y)(x+y)$. Now, think about the area model of multiplication to make your picture.)

Solution: The picture below is a square whose sides are all length $x+y$. One way to express the area is $(x+y)(x+y)$, or $(x+y)^2$. Another way is to add up the smaller boxes that comprise the larger box. This gives $x^2 + 2xy + y^2$, so $(x+y)^2 = x^2 + 2xy + y^2$.



University of South Carolina
Math 221: Math for Elementary Educators
Instructor: Austin Mohr
Section 001
Spring 2010

Test 1 (take-home)

[2] 8. Evaluate each of the following expressions using any method. Note that the arithmetic is to be carried out in base 7.

a. $453_7 + 265_7$

Solution:

$$\begin{array}{cccccc} & & & 4 & 5 & 3_7 \\ + & & & 2 & 6 & 5_7 \\ \hline & & & 6 & 0 & 0_7 \\ & & & 1 & 4 & 0_7 \\ + & & & & 1 & 1_7 \\ \hline & & 1 & 0 & 5 & 1_7 \end{array}$$

b. $514_7 - 355_7$

Solution: If we add 12_7 to both numbers, the problem becomes $526_7 - 400_7$, which is easily evaluated as 126_7.

c. $45_7 \times 56_7$

Solution: The multiplication can be performed in any order. I will do it the same way as in the standard algorithm.

$$\begin{array}{cccccc} & & & & 4 & 5_7 \\ \times & & & & 5 & 6_7 \\ \hline & & & & 4 & 2_7 \\ & & & 3 & 3 & 0_7 \\ & & & 3 & 4 & 0_7 \\ + & & 2 & 6 & 0 & 0_7 \\ \hline & & 3 & 6 & 4 & 2_7 \end{array}$$

d. $546_7 \div 12_7$

<u>Solution</u>: For the problem below, all numbers are assumed to be in base 7. (Please forgive the poor typesetting.)

			4	3
12	|	5	4	6
	-	5	1	
			3	6
	-		3	6
				0