University of South Carolina
Math 115: Precalculus
Instructor: Austin Mohr
Section 006
Fall 2009

Test 4

Do not write on this page. Instead, use the blank paper provided to show all your work and answers. Credit will not be given if no work is shown.

1. Word Problem

Your instructor uncovers an ancient tome with a ritual for achieving blood alcohol content of 1.00 - he will become a being of pure alcohol. The inscription is translated as follows:

The fount of transcendence brims with vodka and tequila.
The total number of shots is to be 100.
Each shot of vodka will measure 45 mL.
Each shot of tequila will measure 55 mL.
The total volume of all drinks consumed will be 5150 mL.

How many shots of vodka and how many shots of tequila must he consume in order to free himself from the prison of blood and bone?

Solution: The cryptic message in the tome defines a system of 2 equations in 2 unknowns. Let v be the number of vodka shots and t be the number of tequila shots.

$$\begin{aligned} v + t &= 100 \\ 45v + 55t &= 5150 \end{aligned}$$

To solve this, we can multiply the first equation by -45 and then add the equations together to get

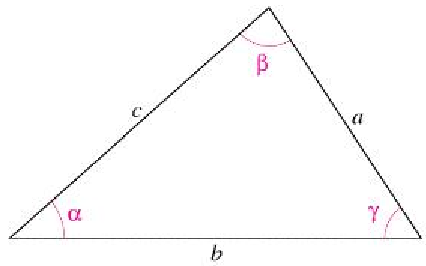
$$\begin{aligned} 10t &= 650 \\ t &= 65. \end{aligned}$$

Plugging this back into the first equation gives

$$\begin{aligned} v + 65 &= 100 \\ v &= 35. \end{aligned}$$

This means that 35 vodka shots and 65 tequila shots are required to cast *power word: kill*.

2. Triangles



a. Solve the triangle above given that $\alpha = 39.5°$, $b = 5.8$, and $c = 3.6$.

Solution: We can solve for a using the Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$a^2 = 5.8^2 + 3.6^2 - 2 \cdot 5.8 \cdot 3.6\cos(39.5°)$$
$$a^2 = 14.38$$
$$a = 3.79$$

Look at the three sides we have. If the triangle has an obtuse angle, it must be β since it is opposite the longest side. So, we will solve for γ first so we don't have to worry about whether β is obtuse or not. Using the Law of Cosines,

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$$
$$\frac{\sin(39.5°)}{3.79} = \frac{\sin\gamma}{3.6}$$
$$.168 = \frac{\sin\gamma}{3.6}$$
$$.605 = \sin\gamma$$
$$\sin^{-1}(.605) = \gamma$$
$$37.21 = \gamma.$$

Finally, we can find β using the fact that every triangle has a total of 180°.

$$180° = 39.5° + \beta + 37.21°$$
$$103.29 = \beta$$

b. How many triangles can be formed if $\alpha = 38°$, $a = 4.7$, and $c = 5.9$? How do you know?

Solution: To do this problem, we need to know the height of the triangle.

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}}$$
$$\sin(38°) = \frac{h}{5.9}$$
$$.616 = \frac{h}{5.9}$$
$$3.63 = h$$

Since side a is longer than the height but shorter than side c, we can form two triangles (one where β will be obtuse and one where β will be acute).

3. Partial Fractions

a. Find the partial fraction decomposition for $\frac{2x-3}{(x-1)(x+3)}$.

Solution: We need to solve

$$\frac{2x-3}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$
$$2x-3 = A(x+3) + B(x-1) \qquad \text{(multiply both sides by (x-1)(x+3))}$$

This equation must be true for *all* x. In particular, see what happens for $x = -3$.

$$-9 = -4B$$
$$\frac{9}{4} = B$$

We can also try $x = 1$.

$$-1 = 4A$$
$$\frac{-1}{4} = A$$

So, the partial fraction decomposition is $\frac{-1}{4(x-1)} + \frac{9}{4(x+3)}$.

b. Set up but do not solve the partial fraction decomposition for $\frac{4x^2-4x-4}{(x+1)^2(x-1)^3}$.

Solution: Every lower power of the factors in the denominator must occur, so we have

$$\frac{4x^2-4x-4}{(x+1)^2(x-1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}.$$

c. Set up but do not solve the partial fraction decomposition for $\frac{x^3+2x^2+3x+3}{(x+1)(x^3+2x^2+2x+2)}$.
(Hint: The polynomial $x^3 + 2x^2 + 2x + 2$ is irreducible.)

Solution: Since we have a high-degree irreducible polynomial as one of the factors, the numerator needs to be one degree less. This gives us

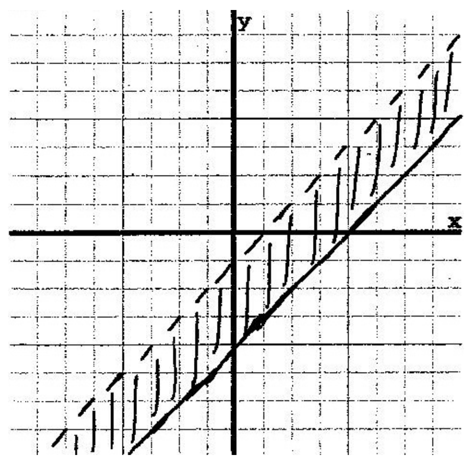
$$\frac{x^3+2x^2+3x+3}{(x+1)(x^3+2x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx^2+Cx+D}{x^3+2x^2+2x+2}.$$

4. Systems of Inequalities

a. Graph the solution to the system of inequalities.

$$x - y < 1$$
$$y + 4 \leq x$$

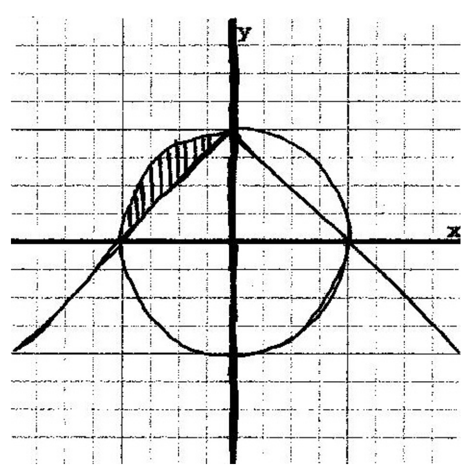
Solution: This one is probably easiest to just solve for y and graph using slope-intercept form.



b. Graph the solution to the system of inequalities.

$$x^2 + y^2 \leq 1$$
$$y \geq -|x| + 1$$
$$x \leq 0$$

Solution: Resist the urge to solve for y in the first equation. Instead, notice that its graph is a circle with center $(0, 0)$ and radius 1. The second graph is $|x|$ (the "V"-shaped graph) with a vertical flip and a vertical shift up 1. The third graph is the horizontal line $x = 0$. Graph all these and do the appropriate shading based on test points.



5. Systems of Equations

a. Solve the system of equations. Your answer will not be unique.

$$x^2 + y^2 = 5$$
$$4x^2 + 9y^2 = 35$$

Solution: One way to solve this is to multiply the first equation by -4 and then add the result to the second equation.

$$5y^2 = 15$$
$$y^2 = 3$$
$$y = \pm\sqrt{3}$$

So, there are two possibilities for y. Let's use each of them separately in the first equation. If $y = \sqrt{3}$, then

$$x^2 + (\sqrt{3})^2 = 5$$
$$x^2 + 3 = 5$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

So, when $y = \sqrt{3}$, we get two possibilities for x. This means that $(-\sqrt{2}, \sqrt{3})$ and $(\sqrt{2}, \sqrt{3})$ are both solutions. We should also see what happens when $y = -\sqrt{3}$. Plug it into the first equation just like before and you'll get $x = \pm\sqrt{2}$. This means that $(-\sqrt{2}, -\sqrt{3})$ and $(\sqrt{2}, -\sqrt{3})$ are also solutions.

b. Solve the system of equations. Your answer will not be unique.

$$x + y + z = 1$$
$$x - y - z = 3$$
$$3x + y + z = 5$$

Solution: Adding equation 1 and 2 gives

$$2x = 4$$
$$x = 2.$$

Since x must be 2, we can go ahead and rewrite all three equations using this fact. Notice that when we do this, every equation turns into $y + z = -1$. This tells us that there won't be a unique solution. Imagine how many pairs of numbers there are that sum to -1. (1 and -2, 2 and -3, 3 and -4, and so on forever). The best we can do is solve for one of the variables in terms of the other. I'll choose z, so I get $z = -1 - y$. This means my solution set is all points of the form $(2, y, -1 - y)$ for any real number y. This says that x must always be 2, y can be whatever I want, and z will be -1 minus whatever y I just chose.

Extra Credit

Solve the system of equations. Your answer will be unique.

(Hint: The addition and substitution methods will take way too long. We did one example of another way to solve systems of equations.)

$$\begin{aligned} -w + x - y + z &= 2 \\ -w - x + 2y - z &= -1 \\ w + 2x - y - z &= 4 \\ -3w + x + 3y - 2z &= 6 \end{aligned}$$

Solution: We can represent this system with matrices.

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 2 & -1 \\ 1 & 2 & -1 & -1 \\ -3 & 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 2 & -1 \\ 1 & 2 & -1 & -1 \\ -3 & 1 & 3 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 2 & -1 \\ 1 & 2 & -1 & -1 \\ -3 & 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 2 & -1 \\ 1 & 2 & -1 & -1 \\ -3 & 1 & 3 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \\ 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}$$

So, $w = 1$, $x = 4$, $y = 3$, and $z = 2$. Note that this works *only* for systems of linear equations (polynomials where all the exponents are 1) and only if the solution is unique.