University of South Carolina
Math 115: Precalculus
Instructor: Austin Mohr
Section 006
Fall 2009

Test 3

Do not write on this page. Instead, use the blank paper provided to show all your work and answers. Credit will not be given if no work is shown.

1. Talking Cat Island

Your plane lands on-time at Talking Cat Island International Airport. Her Excellency the Prime Minister Mittens greets you with a warm nuzzle, as is the custom here. Getting straight to business, she presents you with data regarding the milk supply crisis on Talking Cat Island. Experts estimate the current milk supply (in thousands of gallons) to be given by $f(t) = 350 \cdot .75^t$, where t denotes the number of years from today.

a. How many gallons of milk are currently possessed by the residents of Talking Cat Island?

Solution: Since t measures the number of years from today, the question is telling us that $t = 0$. Plugging this into the function gives

$$\begin{aligned} f(0) &= 350 \cdot .75^0 \\ &= 350. \end{aligned}$$

This means that there are currently 350 thousand gallons of milk.

b. How many gallons will remain at this time next year?

Solution: This is just like the previous problem, except that one year has passed, so $t = 1$. The function tells us

$$\begin{aligned} f(0) &= 350 \cdot .75^1 \\ &= 262.5. \end{aligned}$$

This means that there are currently 262.5 thousand gallons of milk.

c. How many years will it take before the milk levels drop to less than 10,000 gallons? (Trying values until you get it right will not earn you credit. You must use a more sophisticated method.)

Solution: In this problem, we are given the number of gallons and are asked to find t. Note that, if we have 10 thousand gallons, the output of the function is 10 (not 10,000). This means we need to solve

$$\begin{aligned}
10 &= 350 \cdot .75^t \\
.029 &= .75^t \\
\log(.029) &= \log(.75^t) && \text{(take log of both sides)} \\
-1.54 &= t \cdot \log(.75) && (\log(A^p) = p \cdot \log A) \\
-1.54 &= -.125t \\
12.32 &= t.
\end{aligned}$$

This means that in 12.32 years, there will be 10 thousand gallons of milk on the Island. Since the question asks when the milk levels will be *less than* 10 thousand gallons, you might round up to $t = 12.33$ or simply say that any number of years greater than 12.32 will do.

2. Logarithmic and Trigonometric Identities

a. Contract as much as possible.

$$\frac{\frac{1}{2}\log w + \log x}{3\log(y+1) - \log y}$$

Solution:

$$\begin{aligned}
\frac{\frac{1}{2}\log w + \log x}{3\log(y+1) - \log y} &= \frac{\log\left(w^{\frac{1}{2}}\right) + \log x}{\log\left((y+1)^3\right) - \log y} \\
&= \frac{\log\left(w^{\frac{1}{2}}x\right)}{\log\left(\frac{(y+1)^3}{y}\right)}
\end{aligned}$$

Notice that there is no rule to simplify $\frac{\log M}{\log N}$ (if you think there is, read your logarithm rules more closely), so we cannot simplify any further.

b. Show that the following equation is an identity.

$$\frac{\sin^4 x + \sin^2 x \cos^2 x}{\cos^2 x} = \tan^2 x$$

Solution:

$$\begin{aligned}
\frac{\sin^4 x + \sin^2 x \cos^2 x}{\cos^2 x} &= \frac{\sin^2 x(\sin^2 x + \cos^2 x)}{\cos^2 x} \\
&= \frac{\sin^2 x}{\cos^2 x} && (\sin^2 x + \cos^2 x = 1) \\
&= \tan^2 x
\end{aligned}$$

c. Show that the following equation is an identity.

$$\frac{\cos(-x)}{1-\sin x} = \frac{1-\sin(-x)}{\cos x}$$

Solution:

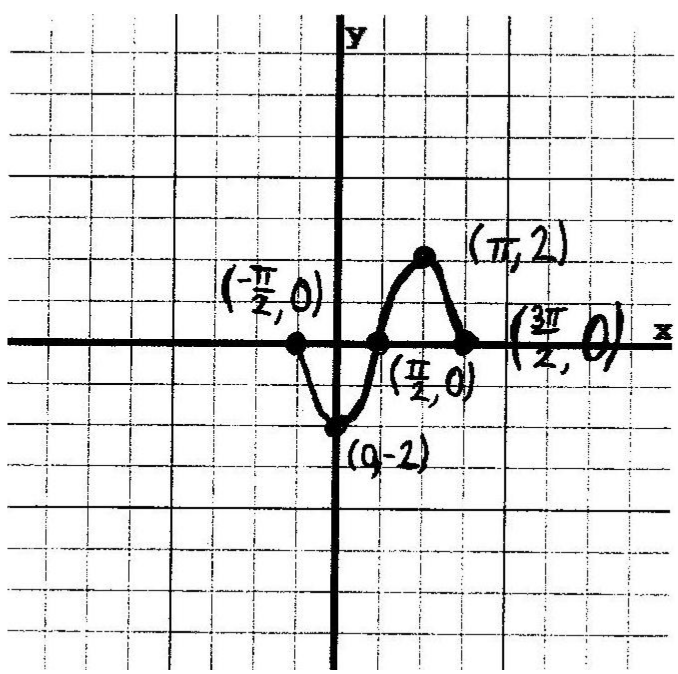
$$\begin{aligned}
\frac{\cos(-x)}{1-\sin x} &= \frac{1-\sin(-x)}{\cos x} \\
\frac{\cos x}{1-\sin x} &= \frac{1+\sin x}{\cos x} && \text{(even/odd identities)} \\
\frac{\cos x}{1-\sin x}(1-\sin x)\cos x &= \frac{1+\sin x}{\cos x}(1-\sin x)\cos x \\
\cos^2 x &= (1+\sin x)(1-\sin x) \\
\cos^2 x &= 1-\sin^2 x \\
\sin^2 x+\cos^2 x &= 1 \\
1 &= 1
\end{aligned}$$

Read up.

3. Graphs of Trigonometric Functions

a. Sketch one cycle of the graph of $-2\sin\left(x+\frac{\pi}{2}\right)$. Give the coordinates of one hill, one valley, and one x-intercept.

Solution: Using what we know about transformations (of any graph, not just trigonometric ones), we see that this is the graph of $\sin x$ shifted to the left by $\frac{\pi}{2}$, stretched vertically by 2, and flipped vertically. Applying these transformations one by one gets us the following graph.

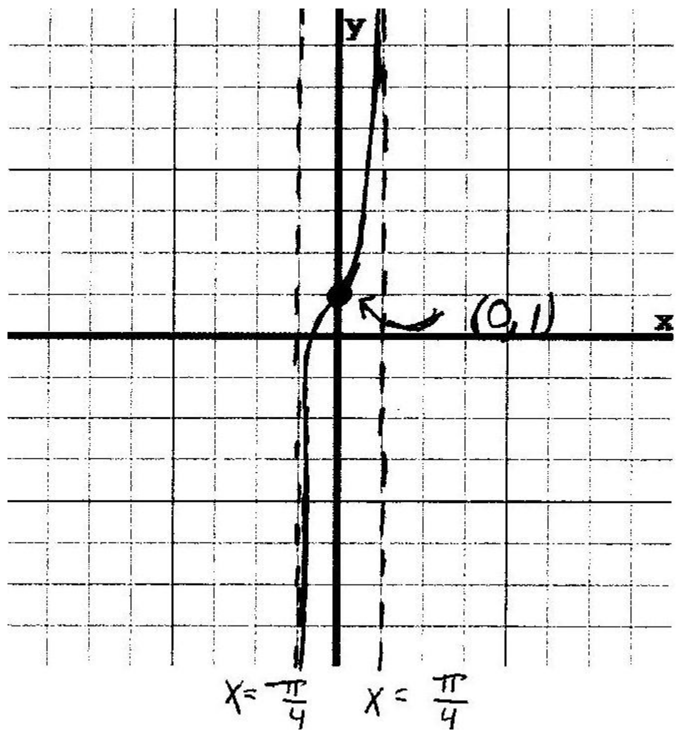


b. Sketch one cycle of the graph of $\tan(2x)+1$. Give the equations of the two asymptotes closest to your cycle and give the coordinates of the y-intercept.

Solution: This is the graph of $\tan x$ with a period change and a vertical shift up 1. To determine what the new period should be, use

$$\frac{\text{standard period}}{B} = \text{modified period}$$
$$\frac{\pi}{2} = \text{modified period.}$$

Since the vertical asymptotes govern the period, we must bring them both in equally so that the period will be $\frac{\pi}{2}$. This gives the following graph.



c. Give the equation of a cosine wave with period $\frac{\pi}{4}$, phase shift $\frac{\pi}{12}$, and range $[2, 12]$.

Solution: We need to use the general form $A\cos(B(x - C)) + D$.
Since the period is $\frac{\pi}{4}$, we use the formula

$$\frac{\text{standard period}}{B} = \text{modified period}$$
$$\frac{2\pi}{B} = \frac{\pi}{4}$$
$$8\pi = B\pi$$
$$8 = B.$$

Since the phase shift is $\frac{\pi}{12}$, we just take $C = \frac{\pi}{12}$ (there is no work to do).

We use the range to tell us both the amplitude A and the vertical shift D. Remember that amplitude is half the distance between a hill and a valley, so this gives

$$\begin{aligned} A &= \frac{1}{2}(12-2) \\ &= 5. \end{aligned}$$

Now, think about the range of $5\cos x$ (the phase shift and period have no bearing on the range, so we don't care about them at the moment). The original $\cos x$ has range $[-1,1]$, so a vertical stretch by 5 will give us a range of $[-5,5]$. The range we *want* however, is $[2,12]$. The only option we have left is a vertical shift, so need to use that somehow. Notice that the current (that is, with no shift) high point is 5, but it should be 12. So, we choose a vertical shift of 7. This also moves the current low point of -5 to 2 where it belongs.

Putting this all together, we get the equation $5\cos(8(x-\frac{\pi}{12}))+7$.

4. <u>Logarithmic and Trigonometric Equations</u>

a. Solve for x. Your answer should be exact.

$$\log(10x+9) - \log x = 2$$

<u>Solution</u>:

$$\begin{aligned} \log(10x+9) - \log x &= 2 \\ \log\left(\frac{10x+9}{x}\right) &= 2 \\ 10^{\log\left(\frac{10x+9}{x}\right)} &= 10^2 && (10^{\text{LHS}} = 10^{\text{RHS}}) \\ \frac{10x+9}{x} &= 100 \\ 10x+9 &= 100x \\ 9 &= 90x \\ \frac{1}{10} &= x \end{aligned}$$

b. Solve for x. You may give either an exact or an approximate answer.

$$5^{x+1} = 3^x$$

<u>Solution</u>:

$$\begin{aligned} 5^{x+1} &= 3^x \\ \log\left(5^{x+1}\right) &= \log\left(3^x\right) \\ (x+1)\log 5 &= x\log 3 \\ x\log 5 + \log 5 &= x\log 3 \\ x\log 5 - x\log 3 &= -\log 5 \\ x(\log 5 - \log 3) &= -\log 5 \\ x &= \frac{-\log 5}{\log 5 - \log 3} && \text{(exact)} \\ x &\approx -3.15 && \text{(approximate)} \end{aligned}$$

c. Find the exact value of $\sin(2\alpha)$ given that $\sin(\alpha) = \frac{2}{5}$ and α is in quadrant II.

Solution: First, use the double-angle formula and work as far as you can.

$$\begin{aligned} \sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ &= 2\left(\frac{2}{5}\right)\cos\alpha \\ &= \frac{4}{5}\cos\alpha \end{aligned}$$

In order to finish the problem, we need to know what $\cos\alpha$ is. The easiest identity that relates $\sin\alpha$ (something you do know) with $\cos\alpha$ (something you want to know) is $\sin^2\alpha+\cos^2\alpha = 1$.

$$\begin{aligned} \sin^2\alpha + \cos^2\alpha &= 1 \\ \left(\frac{2}{5}\right) + \cos^2\alpha &= 1 \\ \frac{4}{25} + \cos^2\alpha &= 1 \\ \cos^2\alpha &= \frac{21}{25} \\ \cos\alpha &= \pm\frac{\sqrt{21}}{5} \end{aligned}$$

Since α is in quadrant II (where values of cosine are always negative), we pick $\cos\alpha = -\frac{\sqrt{21}}{5}$.
Now that we've found $\cos\alpha$, we can finish the original problem.

$$\begin{aligned} \sin(2\alpha) &= \frac{4}{5}\cos\alpha \\ &= \frac{4}{5}\left(-\frac{\sqrt{21}}{5}\right) \\ &= -\frac{4\sqrt{21}}{25} \end{aligned}$$

d. Find all values of x that satisfy. Give an exact answer in radians.

$$\sin^2(3x) = \sin(3x)$$

Solution: Remember that when you are solving for x (in any problem, not just trigonometry problems), you *cannot divide by x*. One of your solutions might be $x = 0$, and so dividing by x is equivalent to dividing by 0, which blows everything to hell. Instead, treat this equation as a quadratic.

$$\begin{aligned} \sin^2(3x) &= \sin(3x) \\ \sin^2(3x) - \sin(3x) &= 0 \\ \sin(3x)(\sin(3x) - 1) &= 0 \end{aligned}$$

This tells us that $\sin(3x) = 0$ or $\sin(3x) - 1 = 0$. We'll solve each one separately.

$$\begin{aligned} \sin(3x) &= 0 \\ \sin^{-1}\sin(3x) &= \sin^{-1} 0 \\ 3x &= k\pi \\ x &= \frac{k\pi}{3} \end{aligned}$$

The next to last line deserves some explanation. If you type $\sin^{-1} 0$ into your calculator, it will only give you one solution for x (namely, 0). Using this and your knowledge of the unit circle, notice that every step of π radians around the circle *also* gives you 0 for the value of sine. This is what it's meant by the phrase $k\pi$ - start at 0 radians and take steps of π around the circle. After that, we just divide by 3 to get x by itself.

The other equation is solved similarly.

$$\begin{aligned} \sin(3x) - 1 &= 0 \\ \sin(3x) &= 1 \\ \sin^{-1}\sin(3x) &= \sin^{-1} 1 \\ 3x &= \frac{\pi}{2} + k(2\pi) \\ x &= \frac{\pi}{6} + \frac{k(2\pi)}{3} \end{aligned}$$

Just like before, your calculator will only give $\sin^{-1} 1 = \frac{\pi}{2}$. Look at the unit circle and notice that the only way to get sine to be 1 is to start at $\frac{\pi}{2}$ and take *full circle* steps. In other words, $\frac{\pi}{2} + k(2\pi)$. To finish, divide both sides by 3 to get x by itself.

Extra Credit

There are seven ways to arrange four labeled balls in two unlabeled buckets so that neither bucket is empty. Show all seven of these arrangements.

Solution:

1 | 234
2 | 134
3 | 124
4 | 123
12 | 34
13 | 24
14 | 23