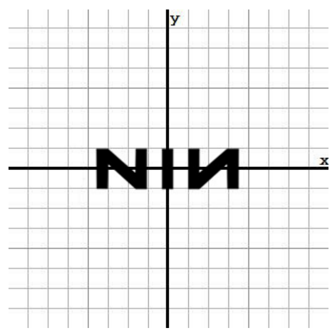
University of South Carolina
Math 115: Precalculus
Instructor: Austin Mohr
Section 006
Fall 2009

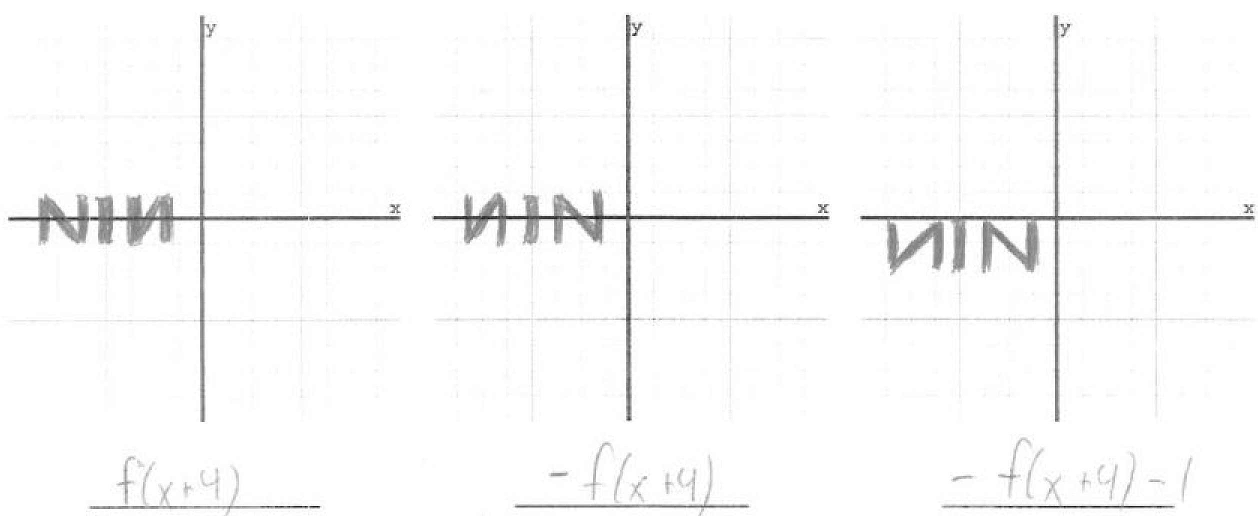
Test 2 Solutions

1. Let $f(x)$ have the following graph:



Sketch the graph of $-f(x+4)-1$ by performing the appropriate translations *one at a time*. At each step, give the function of the graph you are drawing.

Solution: Looking at the function we are supposed to graph, we can tell that there is a horizontal shift by 4 units to the left (the "+ 4" inside the parentheses), a vertical flip (the negative sign next to f), and a vertical shift 1 unit down (the "-1" at the end). I list them in this order because you must perform the transformations in the order of operations. Doing them in this order, your graphs will be:



2. In the following parts, let $f(x) = (x+2)^2(x-1)^3$.

a. What is the y-intercept of $f(x)$?

Solution: The y-intercept of *any* graph (if it has a y-intercept at all) occurs at $x = 0$. Plugging in 0 for x,

$$\begin{aligned} f(0) &= (0+2)^2(0-1)^3 \\ &= -4. \end{aligned}$$

This means that the y-intercept is $(0, -4)$.

b. What is/are the x-intercept(s) of $f(x)$? Describe the behavior of the function at the x-intercepts you find.

Solution: The x-intercepts of *any* graph (if it has any x-intercepts at all) occurs at $f(x) = 0$. This means we need to solve

$$0 = (x+2)^2(x-1)^3$$

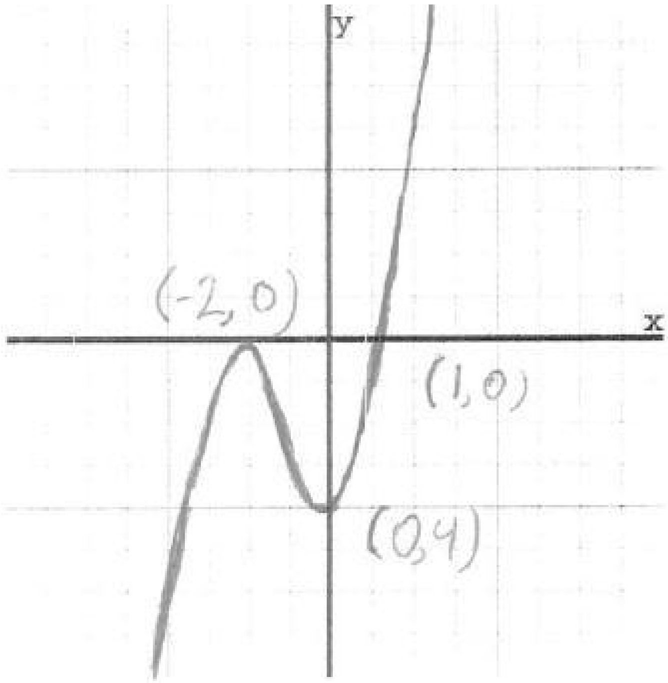
which gives us solutions $x = -2$ and $x = 1$. The solution $x = -2$ comes from the factor $(x+2)^2$, which has even multiplicity, and so the root bounces off the x-axis at this point. The solution $x = 1$ comes from the factor $(x-1)^3$, which has odd multiplicity, and so the root crosses through the x-axis at this point.

c. Describe the end behavior of $f(x)$.

Solution: Imagine multiplying all the terms out. The leading term will be x^5 (you have five sets of parentheses and you can take an x from each one). When you plug positive numbers into the leading term, you get positive outputs. This means that if $x \to \infty$, then $f(x) \to \infty$. When you plug negative numbers into the leading term, you get negative outputs. This means that if $x \to -\infty$, then $f(x) \to -\infty$.

d. Sketch the graph of $f(x)$. Make sure your sketch agrees with the information you found in the previous parts.

Solution: Incorporating all the information from the first three parts gives the following graph.



3. In the following parts, let $f(x) = \frac{2x+1}{x+3}$.
   a. What is the y-intercept of $f(x)$?

   Solution: As before, we plug in 0 for x. This gives us

$$\begin{aligned} f(0) &= \frac{2 \cdot 0 + 1}{0 + 3} \\ &= \frac{1}{3}. \end{aligned}$$

So, the y-intercept is $(0, \frac{1}{3})$.

   b. What is/are the x-intercept(s) of $f(x)$?

   Solution: As before, we set $f(x) = 0$ and solve for x. This gives us

$$\begin{aligned} 0 &= \frac{2x+1}{x+3} \\ 0 &= 2x + 1 \\ \frac{1}{2} &= x \end{aligned}$$

So, the only x-intercept is $(-\frac{1}{2}, 0)$.

   c. What is/are the asymptote(s) of $f(x)$?

Solution: To find vertical asymptotes, check where the function becomes undefined (i.e. where we may potentially divide by zero).

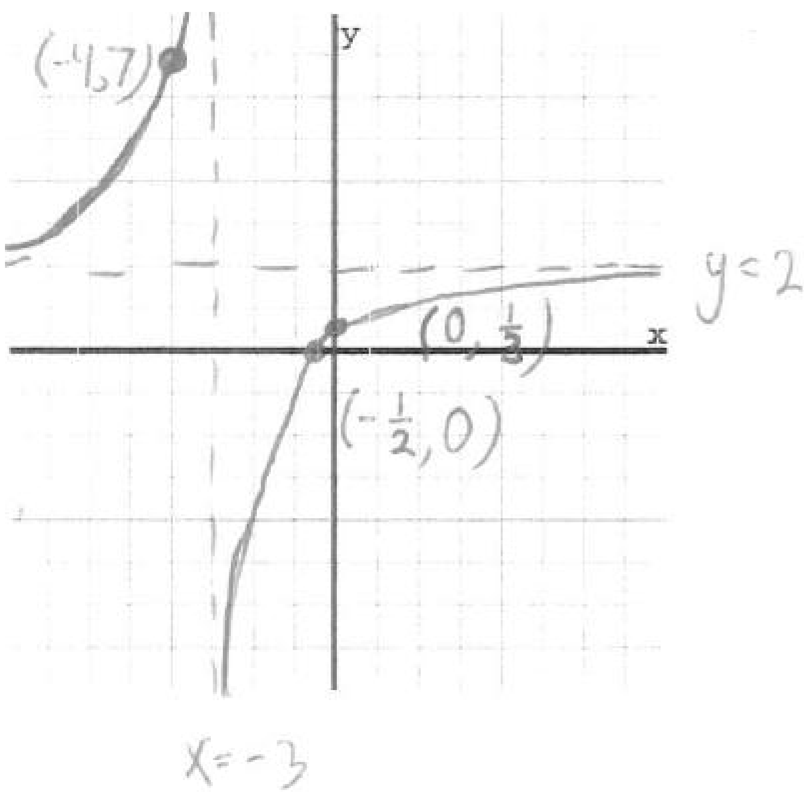
$$x + 3 = 0$$
$$x = -3$$

So, the function has the line $x = -3$ as a vertical asymptote.

To find the horizontal or slant asymptote, compare the degree of the numerator and denominator. They are equal in this case (they are both 1), so that means the ratio of the leading terms gives us the horizontal asymptote. In this case, we have $2x$ and x, so their ratio is 2. So, the graph has the line $y = 2$ as a horizontal asymptote.

d. Sketch the graph of $f(x)$. Make sure your sketch agrees with the information you found in the previous parts.

Solution: Incorporating all the information from the first three parts gives the following graph.



4. In the following parts, let $f(x) = x^2 + 3x$ and $g(x) = x^3 + 5$. Evaluate each expression.

a. $\left(\frac{f}{g}\right)(-2)$

Solution:

$$\begin{aligned}\left(\frac{f}{g}\right)(-2) &= \frac{f(-2)}{g(-2)}\\ &= \frac{(-2)^2+3(-2)}{(-2)^3+5}\\ &= \frac{-2}{-3}\\ &= \frac{2}{3}\end{aligned}$$

b. $f(x+h)-f(x)$

Solution:

$$\begin{aligned}f(x+h)-f(x) &= ((x+h)^2+3(x+h))-(x^2+3x)\\ &= (x^2+2xh+h^2+3x+3h)-(x^2+3x)\\ &= 2xh+h^2+3h\end{aligned}$$

c. $(f\circ g)(x)$

Solution:

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(x^3+5)\\ &= (x^3+5)^2+3(x^3+5)\\ &= (x^6+10x^3+25)+(3x^3+15)\\ &= x^6+13x^3+40\end{aligned}$$

d. $g^{-1}(x)$

Solution: To find the inverse of a function, exchange x and y and solve for y.

$$\begin{aligned}x &= y^3+5\\ x-5 &= y^3\\ \sqrt[3]{x-5} &= y\end{aligned}$$

So, $g^{-1}(x)=\sqrt[3]{x-5}$. You can check that this is indeed the inverse by checking that $g^{-1}(g(x))=x$.

5. Solve each expression for x.
   a. $(x-1)^{-\frac{3}{2}}=27$

Solution:

$$\begin{aligned}
(x-1)^{-\frac{3}{2}} &= 27 \\
((x-1)^{-\frac{3}{2}})^{-\frac{2}{3}} &= 27^{-\frac{2}{3}} \\
x-1 &= \frac{1}{(27^{\frac{1}{3}})^2} \\
x-1 &= \frac{1}{9} \\
x &= \frac{10}{9}
\end{aligned}$$

b. $\sqrt{3x+1} - \sqrt{x} = 1$

Solution:

$$\begin{aligned}
\sqrt{3x+1} - \sqrt{x} &= 1 \\
\sqrt{3x+1} &= 1 + \sqrt{x} \\
(\sqrt{3x+1})^2 &= (1+\sqrt{x})^2 \\
3x+1 &= 1 + 2\sqrt{x} + (\sqrt{x})^2 \\
3x+1 &= 1 + 2\sqrt{x} + x \\
2x &= 2\sqrt{x} \\
x &= \sqrt{x} \\
x^2 &= (\sqrt{x})^2 \\
x^2 &= x \\
x^2 - x &= 0 \\
x(x-1) &= 0 \\
x &= 0 \text{ or } 1
\end{aligned}$$

c. $\frac{x^2-9}{x+1} \geq 0$

Solution: To begin this problem, start by finding all roots and vertical asymptotes of the function (since these are the moments when the output of a rational function can switch signs).

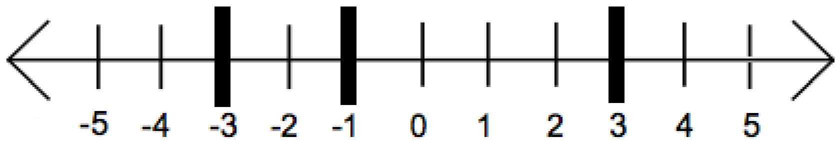
To find the roots, solve

$$\begin{aligned}
\frac{x^2-9}{x+1} &= 0 \\
x^2 - 9 &= 0 \\
x^2 &= 9 \\
x &= -3 \text{ or } 3.
\end{aligned}$$

To find the vertical asymptotes, solve

$$\begin{aligned} x + 1 &= 0 \\ x &= -1. \end{aligned}$$

Plot these three answers on the number line.



This picture breaks up the number line into four intervals. To determine what numbers satisfy the inequality, we just need to see how the function behaves (i.e. is it greater than or equal to 0) at *any* point in these intervals. We'll also check the endpoints of each interval.

$$\begin{aligned}
f(-4) &= \frac{(-4)^2-9}{-4+1} \\
&= -\frac{7}{3} \\
&\not\geq 0 \\
f(-3) &= \frac{(-3)^2-9}{-3+1} \\
&= 0 \\
&\geq 0 \\
f(-2) &= \frac{(-2)^2-9}{-2+1} \\
&= 5 \\
&\geq 0 \\
f(-1) &= \frac{(-1)^2-9}{-1+1} \\
&\text{is undefined} \\
f(0) &= \frac{0^2-9}{0+1} \\
&= -9 \\
&\not\geq 0 \\
f(3) &= \frac{3^2-9}{3+1} \\
&= 0 \\
&\geq 0 \\
f(4) &= \frac{4^2-9}{4+1} \\
&= \frac{7}{5} \\
&\geq 0
\end{aligned}$$

Combining all this information together, the set that satisfies the inequality is $[-3,-1) \cup [3,\infty)$.

6. Your instructor, convinced by a green fairy that he is capable of flight, attempts to leap from the edge of his roof to his neighbor's roof. The function $h(t) = 10 + 10t - 9.8t^2$ models his height (in feet) above the ground t seconds after lift-off.

a. How tall is his roof?

Solution: Ask yourself, "If Austin is teetering precariously on the edge of the roof, do I know anything about t or $h(t)$?". Since t measures the number of seconds *after* the jump and the jump has not yet been made, it must be that $t = 0$. You can now use the function to determine the height at this time.

$$\begin{aligned} h(0) &= 10 + 10 \cdot 0 - 9.8 \cdot 0^2 \\ &= 10 \text{ feet} \end{aligned}$$

b. When does he reach the pinnacle of the jump? How high above the ground is he at this moment?

Solution: Think about the shape of the graph of h. It is a quadratic (since the highest exponent is 2), so it is a parabola. We also know that, since the coefficient of the t^2 term is negative, it is a parabola opening downward. So, the maximum height will be found at the vertex. Using the vertex formula, we find that the maximum height occurs after

$$\begin{aligned} t &= \frac{-b}{2a} \\ &= \frac{-10}{2 \cdot (-9.8)} \\ &\approx .51 \text{ seconds.} \end{aligned}$$

We also want to know how high above the ground I am at this moment. This is found by plugging in the value we have for time:

$$\begin{aligned} h(.51) &= 10 + 10 \cdot .51 - 9.8 \cdot (.51)^2 \\ &\approx 12.55 \text{ feet} \end{aligned}$$

c. The neighbor's roof is way too far away. When will he land instead in the neighbor's rose garden?

Solution: Ask yourself, "If Austin's twisted form is lying on the ground in an alcohol-induced stupor, do I know anything about t or $h(t)$?". Since $h(t)$ measures height *above the ground* and our subject is lying on the ground, it must be that $h(t) = 0$. This means we need to solve

$$0 = 10 + 10t - 9.8t^2.$$

Factoring will be impossible here, so we'll have to resort to the quadratic formula.

$$
\begin{aligned}
t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-10 \pm \sqrt{10^2 - 4 \cdot (-9.8) \cdot 10}}{2 \cdot (-9.8)} \\
&= \frac{-10 \pm \sqrt{492}}{-19.6} \\
&\approx \frac{-10 \pm 22.18}{-19.6} \\
&= \frac{-10 + 22.18}{-19.6} \text{ and } \frac{-10 - 22.18}{-19.6} \\
&\approx -.62 \text{ and } 1.64
\end{aligned}
$$

The negative solution is meaningless here (what does -.62 seconds mean?), so we only want the positive answer. This means that the object of our study crashes into the ground 1.64 seconds after jumping.

Extra Credit: Give a formal proof that $\frac{x+1}{x+2}$ is one-to-one.

Solution: Remember that one-to-one means that if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. This statement is somewhat difficult to work with, so we use the contrapositive (flip the direction of the statement and put "not" in front of both parts): If $f(x_1) = f(x_2)$, then $x_1 = x_2$. To show this is true, start by assuming that $f(x_1) = f(x_2)$ and follow your nose.

$$
\begin{aligned}
f(x_1) &= f(x_2) \\
\frac{x_1 + 1}{x_1 + 2} &= \frac{x_2 + 1}{x_2 + 2} \\
(x_1 + 1)(x_2 + 2) &= (x_2 + 1)(x_1 + 2) && \text{(cancel the denominators)} \\
x_1 x_2 + 2x_1 + x_2 + 2 &= x_1 x_2 + 2x_2 + x_1 + 2 \\
x_1 &= x_2
\end{aligned}
$$

This means that $\frac{x+1}{x+2}$ really is one-to-one.