University of South Carolina
Math 115: Precalculus
Instructor: Austin Mohr
Section 006
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Test 1

Do not write on this page. Instead, use the blank paper provided to show all your work and answers. Keep this page when you are finished.

1. Give the domain of the following functions.
   a. $f(x) = \sqrt{2x-3}$

Solution: We can take the square root of anything greater than or equal to 0 with no problem. If we try to take the square root of a negative number, then we have a problem. So, if we want to make sure that the thing under the square root is greater than or equal to 0, we want

$$\begin{aligned} 2x-3 &\geq 0 \\ 2x &\geq 3 \\ x &\geq \frac{3}{2}. \end{aligned}$$

So, the domain is $x \geq \frac{3}{2}$. In other words, if we choose x to be greater than or equal to $\frac{3}{2}$, then we can take the square root (instead of shrugging our shoulders).

   b. $f(x) = x^2 + 1$

Solution: The two problem cases for us are division by zero and square root of negatives. There is no division and no square root in this function, so that means any x is ok. So, the domain is all real numbers.

   c. $f(x) = \frac{2x-1}{x^2+14x-15}$

Solution: The potential problem here is that we might choose a bad x and end up dividing by zero. Let's try to find out which x's are bad. To do this, we should solve

$$\begin{aligned} x^2 + 14x - 15 &= 0 \\ (x+15)(x-1) &= 0 \qquad \text{(factor the quadratic)} \\ x + 15 = 0 &\text{ or } x - 1 = 0 \\ x = -15 &\text{ or } x = 1. \end{aligned}$$

Take a step back and think about what we've done. We just found all the x's to *do* make the denominator zero. This is the thing we're trying to avoid, so we should say that the domain is all x's *except* -15 and 1. Notice that the numerator played absolutely no part in this.

2. Write in simplest terms. Your answer should not contain negative exponents or radicals.
   a. $\frac{3}{x-5} - \frac{2}{x+7}$

Solution: We need to construct a common denominator. The fraction on the left has an $x-5$ term and the fraction on the right does not. So, we should multiply the numerator and denominator of the second fraction by $x-5$:

$$\frac{3}{x-5}-\frac{2(x-5)}{(x-5)(x+7)}.$$

Similarly, the fraction on the left needs a factor of $x+7$, so we do the same thing:

$$\frac{3(x+7)}{(x-5)(x+7)}-\frac{2(x-5)}{(x-5)(x+7)}.$$

Now we have a common denominator, so we can subtract:

$$\begin{aligned}\frac{3(x+7)-2(x-5)}{(x-5)(x+7)}\\ \frac{(3x+21)-(2x-10)}{(x-5)(x+7)}\\ \frac{x+31}{(x-5)(x+7)}\end{aligned}$$

b. $\sqrt{16xy^4} \div \sqrt[3]{\frac{8}{x^5y^2}}$

Solution: Let's simplify each term first, then worry about the division. The first term simplifies to

$$\begin{aligned}\sqrt{16xy^4} &= (16xy^4)^{\frac{1}{2}}\\ &= 16^{\frac{1}{2}}x^{\frac{1}{2}}(y^4)^{\frac{1}{2}}\\ &= 4x^{\frac{1}{2}}y^2.\end{aligned}$$

The second term simplifies to

$$\begin{aligned}\sqrt[3]{\frac{8}{x^5y^2}} &= \left(\frac{8}{x^5y^2}\right)^{\frac{1}{3}}\\ &= \frac{8^{\frac{1}{3}}}{(x^5)^{\frac{1}{3}}(y^2)^{\frac{1}{3}}}\\ &= \frac{2}{x^{\frac{5}{3}}y^{\frac{2}{3}}}.\end{aligned}$$

Now let's return to the original problem:

$$\begin{aligned}\sqrt{16xy^4} \div \sqrt[3]{\frac{8}{x^5y^2}} &= 4x^{\frac{1}{2}}y^2 \div \frac{2}{x^{\frac{5}{3}}y^{\frac{2}{3}}}\\ &= 4x^{\frac{1}{2}}y^2 \cdot \frac{x^{\frac{5}{3}}y^{\frac{2}{3}}}{2}\\ &= \frac{(4x^{\frac{1}{2}}y^2)(x^{\frac{5}{3}}y^{\frac{2}{3}})}{2}\\ &= 2x^{\frac{13}{6}}y^{\frac{8}{3}}.\end{aligned}$$

c. $\frac{\left(x^{-3}y^2\right)^{-1}}{x^{-2}y^6}$

Solution: This problem is just about applying the exponent rules carefully.

$$\begin{aligned}\frac{\left(x^{-3}y^{2}\right)^{-1}}{x^{-2}y^{6}} &= \frac{x^{3}y^{-2}}{x^{-2}y^{6}} \\ &= \frac{x^{3}x^{2}}{y^{6}y^{2}} \\ &= \frac{x^{5}}{y^{8}}\end{aligned}$$

3. Solve for x.

a. $|1-2x| \le 5$

Solution: See "Further Ramblings on Absolute Value Inequalities" for a more detailed description. The short answer is that, since the absolute value is less than or equal to some value, we want to solve

$$-5 \le 1-2x \text{ and } 1-2x \le 5.$$

The first inequality gives

$$\begin{aligned}-5 &\le 1-2x \\ -5+2x &\le 1 \\ 2x &\le 6 \\ x &\le 3.\end{aligned}$$

The second inequality gives

$$\begin{aligned}1-2x &\le 5 \\ 1 &\le 5+2x \\ -4 &\le 2x \\ -2 &\le x.\end{aligned}$$

Putting this together, we have

$$-2 \le x \text{ and } x \le 3$$

which can also be written as $-2 \le x \le 3$ or in interval notation as $[-2, 3]$.

b. $x = \frac{-2}{x+3}$

Solution: Since we are actually solving for x here (rather than just simplifying, like in problem 2a), we can try to cancel denominators.

$$\begin{aligned}x &= \frac{-2}{x+3} \\ x(x+3) &= \frac{-2}{x+3} \cdot (x+3) \\ x(x+3) &= -2\end{aligned}$$

Now that the denominator is gone, you can think about how to proceed. Notice that, if you were to distribute on the left, you would have a term of x^2, so trying to factor or use the quadratic formula is probably a good way to go. In order to do that, though, we need everything on one side of the equation.

$$\begin{aligned} x(x+3) &= -2 \\ x^2+3x &= -2 \\ x^2+3x+2 &= 0 \\ (x+1)(x+2) &= 0 \\ x+1=0 &\text{ or } x+2=0 \\ x=-1 &\text{ or } x=-2 \end{aligned}$$

c. $x^3+6x^2+11x+6=0$ given that one of the factors is $x+2$

<u>Solution</u>: The fact that $x+2$ is a factor means that we should be able to divide $x^3+6x^2+11x+6$ by $x+2$ and get no remainder.

$$\begin{array}{r} x^2 \quad +4x+3 \\ x+2\overline{)\ x^3+6x^2+11x+6} \\ -x^3-2x^2 \qquad\qquad \\ \hline 4x^2+11x \quad \\ -4x^2 \quad -8x \quad \\ \hline 3x+6 \\ -3x-6 \\ \hline 0 \end{array}$$

Now we know that

$$x^3+6x^2+11x+6=(x+2)(x^2+4x+3)$$

We can further factor the quadratic part to get

$$x^3+6x^2+11x+6=(x+2)(x+1)(x+3).$$

Returning to the original question

$$\begin{aligned} x^3+6x^2+11x+6 &= 0 \\ (x+1)(x+2)(x+3) &= 0 \\ x+1=0 \text{ or } x+2=0 &\text{ or } x+3=0 \\ x=-1 \text{ or } x=-2 &\text{ or } x=-3. \end{aligned}$$

4. In each part, give the equation of the function described.

a. The line through the points $(-1,-2)$ and $(-5,1)$

<u>Solution</u>: Every line looks like $y=b+mx$, where b is the y-intercept and m is the slope. We can use the slope formula to find the slope:

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{1-(-2)}{-5-(-1)} \\ &= -\frac{3}{4} \end{aligned}$$

We can update the equation to

$$y = b - \frac{3}{4}x.$$

Since we have x, y, and a single unknown b, we can plug in a point and solve for b.

$$\begin{aligned} -2 &= b - \frac{3}{4}(-1) \\ -2 &= b + \frac{3}{4} \\ -\frac{11}{4} &= b \end{aligned}$$

So, the whole equation is $y = -\frac{11}{4} - \frac{3}{4}x$.

b. The circle with center $(1, -3)$ through the point $(4, 2)$
Solution: Every circle looks like

$$(x - h)^2 + (y - k)^2 = r^2$$

where (h, k) is the center of the circle and r is the radius. We're given the center, so we can already write

$$(x - 1)^2 + (y + 3)^2 = r^2$$

(be careful about the $y - (-3)$ part). Just like the problem before, we have x, y, and the single unknown value r. Since the point $(4, 2)$ lies on the circle, it must satisfy the equationa above, so we can plug it in and solve for r.

$$\begin{aligned} (4 - 1)^2 + (2 + 3)^2 &= r^2 \\ 3^2 + 5^2 &= r^2 \\ 34 &= r^2 \end{aligned}$$

Notice that there's really no need to take the square root here. We want r^2 in the equation for the circle anyway, so we may as well just leave it as r^2. So, our final answer is

$$(x - 1)^2 + (y + 3)^2 = 34.$$

c. A polynomial with roots $x = -1$, $x = 1$, and $x = 2$ (Hint: Think about how this polynomial would look in factored form.)

Solution: If we want to have the specified roots, then the factored form has to look like

$$(x + 1)(x - 1)(x - 2).$$

Notice that each of the roots corresponds to one factor that it can make zero. (There are actually infinitely-many answers to this question, but this is the simplest one.)

5. After an evening of drunken soul-searching, your instructor realizes his true calling is to tend bars. He purchases a fire-damaged trailer in southern Georgia for $40,000 and opens shop. He estimates he will spend $2,500 monthly on alcohol.

a. Construct a function $f(t)$ that models the total cost to run the bar, where t is measured in months.

Solution: The situation can be described by a linear function, since we start with some fixed cost (the initial value) and then pay a certain fixed amount each month (the rate of change). Since the initial value is \$40,000 and the rate of change is \$2,500, the function should be $f(t) = 40000 + 2500t$.

b. How much money will have been spent over the course of the first year?

Solution: Since 12 months have passed, we want to take $t = 12$ in the function.

$$\begin{aligned} f(12) &= 40000 + 2500 \cdot 12 \\ &= 70000 \end{aligned}$$

This means that \$70,000 is required to run Austin's Country-Style Drinkery™ for the first year.

c. Assume that, instead of serving customers, the owner drinks all the alcohol himself. How long can he continue to run the bar if he only has \$60,000?

Solution: We know that the amount of money spent (the output) is \$60,000, but we don't know how long it takes to spend it (the input). *This means we must be solving for t.* We need to solve

$$\begin{aligned} 60000 &= 40000 + 2500t \\ 20000 &= 2500t \\ 8 &= t \end{aligned}$$

This means that after eight months, the fictitious version of me that has sixty grand to blow on a bar will be forced to beg USC for his old job back.

Extra Credit: Expand $(2x - 3y)^4$ using Pascal's Triangle.

Solution: We need to construct the first five rows of Pascal's Triangle (remember the first row is "Row 0").

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \end{array}$$

Now, recall the "pattern" for expanding binomials.

$$\begin{gathered} 1 \cdot (2x)^4(-3y)^0 + 4 \cdot (2x)^3(-3y) + 6 \cdot (2x)^2(-3y)^2 + 4 \cdot (2x)(-3y)^3 + 1 \cdot (2x)^0(-3y)^4 \\ 16x^4 + 4 \cdot (8x^3)(-3y) + 6 \cdot (4x^2)(9y^2) + 4 \cdot (2x)(-27y^3) + 81y^4 \\ 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4 \end{gathered}$$