Quiz 1 Solutions

1. Find the domain of

$$\frac{3x+5}{2x^3+x^2-5x+2}$$

given that $x-1$ is a factor of the denominator.

Solution: The domain of a function is the set of x-values that you are allowed to plug in to the function. With a rational expression (i.e. a fraction with a polynomial in the top and bottom), the only thing that isn't allowed is division by zero. So, in order to find the domain, we need to find solve

$$2x^3 + x^2 - 5x + 2 = 0$$

When we solve this, we'll know which x-values to throw away (since using these values for x will make the denominator zero, which isn't allowed). Notice that we don't care about the numerator at all.

The only way to solve the equation above is by factoring. We wouldn't normally know how to factor something like $2x^3 + x^2 - 5x + 2$, but we're given a hint: $x - 1$ is a factor of $2x^3 + x^2 - 5x + 2$. This means

$$2x^3 + x^2 - 5x + 2 = (x - 1)(\text{something})$$

To solve for "something", we divide both sides by $x - 1$

$$(2x^3 + x^2 - 5x + 2) \div (x - 1) = \text{something}$$

To figure out the lefthand side, we need to use long division.

$$\begin{array}{r|l} & \quad\quad\quad 2x^2 + 3x - 2 \\ x-1) & 2x^3 \;\; + x^2 - 5x + 2 \\ & -2x^3 + 2x^2 \\ \hline & \quad\quad 3x^2 - 5x \\ & \quad -3x^2 + 3x \\ \hline & \quad\quad\quad -2x + 2 \\ & \quad\quad\quad\;\; 2x - 2 \\ \hline & \quad\quad\quad\quad\quad 0 \end{array}$$

Since the quotient came out with no remainder, we know that

$$2x^3 + x^2 - 5x + 2 = (x - 1)(2x^2 + 3x - 2)$$

Now we can focus on just the $2x^2 + 3x - 2$ part, which is in $ax^2 + bx + c$ form. First, notice that there is no common factor, so we don't need to worry about that. (If there was a common factor, we would have to factor this out before proceeding.) We need factors of -4 (since $a \cdot c = -4$) that sum up to 3 (since $b = 3$). List out the factors of -4 and you will find that the pair 4, -1 has the correct sum. Use these numbers to set up the "box".

	x	2
$2x$	$2x^2$	$4x$
-1	$-x$	-2

This tells us that $2x^2 + 3x - 2 = (x + 2)(2x - 1)$, so all together we have

$$\begin{aligned} 6x^3 + 7x^2 - x - 12 &= (x - 1)(2x^2 + 3x - 2) \\ &= (x - 1)(x + 2)(2x - 1) \end{aligned}$$

The point of all this factoring is that we can now solve

$$(x - 1)(x + 2)(2x - 1) = 0$$

instead of the unfactored version. If the product of a bunch of factors equals zero, then one of the factors must be zero. So, we solve

$$\begin{aligned} x - 1 &= 0 \\ x &= 1 \end{aligned}$$

or

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

or

$$\begin{aligned} 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

What does this tell us? If we try to plug any of these numbers into the original function (the rational expression given in the problem), then the denominator will equal zero and we won't know how to evaulate it (anything divided by zero is undefined). This means that the domain (that is, the numbers we can use) is any number except 1, -2, or $\frac{1}{2}$.

2. Perform the subtraction

$$\frac{2}{2x-3y} - \frac{3x^2-y}{8x^3-27y^3}$$

Your answer will be a single rational expression.

Solution: Whenever you are adding or subtracting fractions (even ones with just numbers), you have to find a common demoninator. For rational expressions, it is easiest to do this when the denominators are factored. The first denominator $(2x - 3y)$ isn't going to factor, since there is no common factor and the exponents on the variables are all one. The second denominator $(8x^3 - 27y^3)$ is a difference of cubes. Recall that (using my "big X" and "big Y" notation),

$$X^3 - Y^3 = (X - Y)(X^2 + XY + Y^2).$$

We want the lefthand side of this equation to be the $8x^3 - 27y^3$, so how should we pick X and Y? We need to solve

$$\begin{aligned} X^3 &= 8x^3 \\ (X^3)^{\frac{1}{3}} &= (8x^3)^{\frac{1}{3}} \\ X &= 8^{\frac{1}{3}}(x^3)^{\frac{1}{3}} \\ X &= 2x \end{aligned}$$

and

$$\begin{aligned} Y^3 &= 27y^3 \\ (Y^3)^{\frac{1}{3}} &= (27y^3)^{\frac{1}{3}} \\ Y &= 27^{\frac{1}{3}}(y^3)^{\frac{1}{3}} \\ Y &= 3y \end{aligned}$$

So, we have

$$\begin{aligned} 8x^3 - 27y^3 &= (2x-3y)((2x)^2 + (2x)(3y) + (3y)^2) \\ &= (2x-3y)(4x^2+6xy+9y^2) \end{aligned}$$

Now that we know how the second denominator factors, let's rewrite the problem.

$$\frac{2}{2x-3y} - \frac{3x^2-y}{(2x-3y)(4x^2+6xy+9y^2)}$$

What do we need to do to get a common denominator? Both denominators already have a $2x-3y$ term in them, but the rational expression on the left is missing the $4x^2+6xy+9y^2$ term. Just like with regular fractions, we need to multiply the numerator and the denominator by this value to get the denominators the same.

$$\begin{aligned} \frac{2}{2x-3y} - \frac{3x^2-y}{(2x-3y)(4x^2+6xy+9y^2)} &= \frac{2}{2x-3y}\cdot\frac{4x^2+6xy+9y^2}{4x^2+6xy+9y^2} - \frac{3x^2-y}{(2x-3y)(4x^2+6xy+9y^2)} \\ &= \frac{2(4x^2+6xy+9y^2)}{(2x-3y)(4x^2+6xy+9y^2)} - \frac{3x^2-y}{(2x-3y)(4x^2+6xy+9y^2)} \\ &= \frac{8x^2+12xy+18y^2}{(2x-3y)(4x^2+6xy+9y^2)} - \frac{3x^2-y}{(2x-3y)(4x^2+6xy+9y^2)} \\ &= \frac{(8x^2+12xy+18y^2)-(3x^2-y)}{(2x-3y)(4x^2+6xy+9y^2)} \\ &= \frac{5x^2+12xy+18y^2+y}{8x^3-27y^3} \end{aligned}$$

3. Simplify

$$\left(\frac{x^{-3}+y^{-2}}{3x^{-3}y^2}\right)^2$$

Your answer should be a single rational expression (which means no negative exponents).

Solution: Using exponent rules, we have

$$\left(\frac{x^{-3}+y^{-2}}{3x^{-3}y^2}\right)^2 = \frac{(x^{-3}+y^{-2})^2}{(3x^{-3}y^2)^2}$$

Notice how the numerator and the denominator differ. The numerator has a plus in it, so we have to use FOIL. The denominator is all multiplication, so we can just distribute the exponent.

$$\begin{aligned} \left(\frac{x^{-3}+y^{-2}}{3x^{-3}y^2}\right)^2 &= \frac{(x^{-3}+y^{-2})^2}{(3x^{-3}y^2)^2} \\ &= \frac{(x^{-3})^2 + x^{-3}y^{-2} + x^{-3}y^{-2} + (y^{-2})^2}{3^2(x^{-3})^2(y^2)^2} \\ &= \frac{x^{-6}+2x^{-3}y^{-2}+y^{-4}}{9x^{-6}y^4} \\ &= \frac{x^6(x^{-6}+2x^{-3}y^{-2}+y^{-4})}{9y^4} \\ &= \frac{x^6x^{-6}+2x^6x^{-3}y^{-2}+x^6y^{-4}}{9y^4} \\ &= \frac{1+2x^3y^{-2}+x^6y^{-4}}{9y^4} \end{aligned}$$

We're almost done, except that the problem specifies that you cannot leave negative exponents in your answer. We cannot just grab the y's and move them down because there is addition involved. Instead, multiply the numerator and denominator by y^4.

$$\begin{aligned}\frac{1+2x^3y^{-2}+x^6y^{-4}}{9y^4}\cdot\frac{y^4}{y^4} &= \frac{(1+2x^3y^{-2}+x^6y^{-4})y^4}{9y^4y^4}\\ &= \frac{y^4+2x^3y^{-2}y^4+x^6y^{-4}y^4}{9y^4y^4}\\ &= \frac{y^4+2x^3y^2+x^6}{9y^8}\end{aligned}$$

This is your final answer as far as I'm concerned. We can actually factor the numerator to get

$$\frac{(x^3+y^2)^2}{9y^8}$$

but it uses a rather complicated substitution, so it's not crucial to understand this.

4. Simplify

$$\sqrt{27x^6}\div\sqrt[3]{x^8}$$

Your answer should be a single radical expression.

Solution: Using the exponent rules, we have

$$\begin{aligned}\sqrt{27x^6}\div\sqrt[3]{x^8} &= (27x^6)^{\frac{1}{2}}\div(x^8)^{\frac{1}{3}}\\ &= (27^{\frac{1}{2}}(x^6)^{\frac{1}{2}})\div(x^8)^{\frac{1}{3}}\\ &= (27^{\frac{1}{2}}x^3)\div x^{\frac{8}{3}}\end{aligned}$$

Before moving on, we should see if the 27 has any perfect squares. Checking the first few numbers, we see that

$$\begin{aligned}27^{\frac{1}{2}} &= (9\cdot 3)^{\frac{1}{2}}\\ &= 9^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}\\ &= 3\cdot 3^{\frac{1}{2}}\end{aligned}$$

Let's use this simpler expression instead.

$$\begin{aligned}(27^{\frac{1}{2}}x^3)\div x^{\frac{8}{3}} &= (3\cdot 3^{\frac{1}{2}}\cdot x^3)\div x^{\frac{8}{3}}\\ &= \frac{3\cdot 3^{\frac{1}{2}}\cdot x^3}{x^{\frac{8}{3}}}\\ &= 3\cdot 3^{\frac{1}{2}}\cdot x^{\frac{1}{3}}\end{aligned}$$

We're almost finished, except that the problem says we must give a single radical as the answer. To do that, we need to convert all the fractional exponents to a common denominator.

$$\begin{aligned}3\cdot 3^{\frac{1}{2}}\cdot x^{\frac{1}{3}} &= 3\cdot 3^{\frac{3}{6}}\cdot x^{\frac{2}{6}}\\ &= 3\cdot(3^3\cdot x^2)^{\frac{1}{6}}\\ &= 3\cdot(27\cdot x^2)^{\frac{1}{6}}\\ &= 3\sqrt[6]{27x^2}\end{aligned}$$