Core Homework

Bring a printed draft to class on the specified due date.

1. (**Draft: March 22, Final March 29**) The *n-dimensional hypercube* Q_n is a simple graph whose vertex set is $\{(x_1, \ldots, x_n) \mid x_i \in \{0,1\}\}$. Two vertices are adjacent in this graph if and only if they agree in exactly $n-1$ coordinates. Prove that Q_n has a Hamiltonian cycle for $n \geq 2$. (Hint: Use induction on n and the fact that Q_n contains two distinct copies of Q_{n-1}.)

2. (**Draft: March 22, Final March 29**) A path P has *maximum length* in a graph G provided G contains no path whose length is strictly greater than that of P. Prove that if a connected graph contains two paths of maximum length, then those paths share at least one vertex. (Hint: Suppose this is not the case and look for a longer path.)

One Per Student

Complete only the problem assigned to you. We should meet briefly at my office on **March 22 at 3 pm** to exchange drafts (remember to bring three printed copies). Please return your critiques to the appropriate person sometime on **March 25**. (I will place my critiques outside my office door that day.) Your revised final version is due **March 29**.

Megan How many *distinct* Hamiltonian cycles does $K_{n,n}$ contain? (Hint: Count every possible cycle and divide away the repetitions resulting from flips and rotations.)

Yesenia Let G be the union of k disjoint cycles, each of length r. How many automorphisms does G have? (Hint: First order the r cycles, then orient each cycle. In how many ways can these actions take place?)

Carter Find the smallest tree with at least one edge that has no non-trivial automorphisms. Prove your tree is indeed the smallest possible. (Hint: Can such a tree exist if all vertices have degree 1 or 2?)

Kenzie A set S of vertices of G is called an *independent set* if no two vertices of S are adjacent in G. For a graph G, let $\alpha(G)$ denote the size of a maximum independent set. A set of vertices T of G is called a *vertex cover* if all edges of G have at least one of their vertices in T. Let $\tau(G)$ be the size of the smallest vertex cover of G. For any simple graph G on n vertices, prove that $\alpha(G) + \tau(G) = n$. (Hint: Prove two separate inequalities: $n - \tau(G) \leq \alpha(G)$ and $n - \alpha(G) \geq \tau(G)$. Combine these to reach the desired conclusion.)