Updated February 17, 2019.

Problem 2

Problem Statement

Let a be any integer and let b and c be odd integers. Prove that $ab + ac$ is an even integer.

Common Feedback

- Avoid the word "prove" in the statement of your proposition. Use a conditional statement, as in the solution below.
- Always define your variables. If you want a to be odd, then write something like $a = 2k + 1$ **for some integer** k.
- As far as possible, avoid working on both sides of the equation in an aligned environment. Rather than an equation with two sides to modifiy, think of an aligned environment as one long string of equalities: this equals this equals this equals ...
- Take care not to accidentally make your variables equal one another. For example, if you say $a = 2k + 1$ and $b = 2k + 1$, then you have inadvertantly claimed that $a = b$.

Solution (from Deanna Meyers)

Proposition. *If a, b, and c are odd integers, $a + bc$ is an even integer*

Proof. Since a, b, and c are odd, we may write $a = 2k + 1$, $b = 2l + 1$ and $c = 2m + 1$. We aim to show that $a + bc$ is even, so we consider

$$\begin{aligned} a + bc &= (2k + 1) + (2l + 1)(2m + 1) \\ &= 2k + 2l + 2m + 4lm + 2 \\ &= 2(k + l + m + lm + 1) \end{aligned}$$

By the closure of the integers under addtion and multiplication, we know that $k + l + m + ml + 1$ is an integer. Call this integer n, so that we have $a + bc = 2n$. Therefore, $a + bc$ is even.

□

Problem 4

Problem Statement

Definition. *We say an integer k **divides** another integer n provided we can write $n = km$ for some integer m. (For example, 3 divides 15 since $15 = 3 \cdot 5$.)*

Show that the following statement is not true in general by writing its negation and giving a specific example where the negation is true.

If a divides bc, then a divides b or a divides c.

Common Feedback

- The negation of a conditional is not another *conditional*. It is a *state of affairs* in which the hypothesis holds and the conclusion does not. In symbols: $\neg(P \to Q) \equiv P \wedge \neg Q$
- Keep an eye out for applications of DeMorgan's Law inside more complex conditionals. In this example, the original statement is of the form $P \to (Q \vee R)$. The negation is therefore $P \wedge \neg(Q \vee R)$. DeMorgan would simplify this to $P \wedge \neg Q \wedge \neg R$.

Solution (from Logan Rowe)

Negation: a divides bc and a doesn't divide b and a doesn't divide c.
Example: $a = 6$, $b = 4$, $c = 3$: $6 \mid 12$ and $6 \nmid 4$ and $6 \nmid 3$

Problem 7

Prove the following statement by contraposition (see exercise # 9 from Section 3.2 in the text for more details):

Let $x \in \mathbb{R}$. If x is irrational, then $\sqrt{x}$ is irrational.

Common Feedback

- Whenever possible, try to avoid making changes to both sides of an equation when carrying out a long string of algebraic manipulations.
- Use `\left(` and `\right)` to get parentheses that stretch to fit your symbols (such as around a fraction).

Solution (from Grace Moravec)

Definition. *A real number x is defined as a rational number, provided there exists integers m and n with $n \neq 0$, such that $x = \frac{m}{n}$.*

Proposition. *For real number x, if x is irrational, then $\sqrt{x}$ is irrational.*

Proof. Taking the contrapositive of the proposition, we assume that if $\sqrt{x}$ is rational, then real number x is rational. Since $\sqrt{x}$ is thereby rational, we can substitute the ratio of integers m and n as defined above for $\sqrt{x}$.
Therefore, we consider:

$$\sqrt{x} = \frac{m}{n}$$

To isolate x, we remove the square root by squaring both sides of the equation.

$$\left(\sqrt{x}\right)^2 = \left(\frac{m}{n}\right)^2$$
$$x = \frac{m^2}{n^2}$$

Since, by the closure of integers under multiplication, the square of an integer is also an integer, we then conclude that m^2 and n^2 are integers. Therefore, real number x is a rational number, as it is equal to the definition of a rational number. Thus, we know the original proposition to be true because the contrapositive, equivalent to the original proposition, is is true. $\square$