Example Proofs

Austin Mohr

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Direct Proof

Proposition. *If n is odd, then n^3 is odd.*

Proof. Since n is odd, we may write $n = 2k+1$ for some integer k. We aim to show that n^3 is odd, so we consider

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1. \end{aligned}$$

By the closure of the integers under addition and multiplication, we know that $4k^3 + 6k^2 + 3k$ is an integer. Call this integer m, so that we have $n^3 = 2m+1$. Therefore, n^3 is odd. □

Weak Induction

Proposition. *For all natural numbers n, we have the identity*

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Proof. (by induction)

First observe that when $n = 1$, the identity becomes

$$1 = \frac{1}{2} \cdot 1 \cdot 2,$$

which is a true statement.

Suppose now that there exists a natural number k such that

$$1 + 2 + \cdots + k = \frac{1}{2}k(k+1).$$

We aim to show that this implies

$$1 + 2 + \cdots + (k+1) = \frac{1}{2}(k+1)(k+2).$$

Applying the inductive hypothesis in the first line, we have

$$\begin{aligned}(1+2+\cdots+k)+(k+1) &= \frac{1}{2}k(k+1)+(k+1)\\ &= (k+1)\left(\frac{1}{2}k+1\right)\\ &= \frac{1}{2}(k+1)(k+2),\end{aligned}$$

as desired. □

Strong Induction

Proposition. *Define the recurrence $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$ with $a_0 = 1$ and $a_1 = 2$. For all natural numbers n, we have the inequality $a_n \leq 3^n$.*

Proof. (by strong induction)

First observe that when $n \in \{1, 2\}$, the inequality becomes $1 \leq 3^0$ and $2 \leq 3^1$, both of which are true statements.

Suppose now that there exists a natural number $k \geq 3$ such that $a_{k-1} \leq 3^{k-1}$ and $a_k \leq 3^k$. We aim to show that these together imply $a_{k+1} \leq 3^{k+1}$. To that end, observe

$$\begin{aligned}a_{k+1} &= 2a_k + a_{k-1} && \text{(by definition)}\\ &\leq 2\cdot 3^k + 3^{k-1} && \text{(by strong inductive hypothesis)}\\ &= 3^{k-1}(2\cdot 3+1)\\ &< 3^{k-1}\cdot 9\\ &= 3^{k+1},\end{aligned}$$

as desired. □

Set Algebra

Proposition. *The identity $(A \cup B) - C = (A - C) \cup (B - C)$ holds for all sets A, B, and C.*

Proof. Observe,

$$\begin{aligned}(A\cup B) - C &= (A \cup B) \cap C^c && \text{(definition of set difference)}\\ &= (A\cap C^c) \cup (B \cap C^c) && \text{(distributive law)}\\ &= (A - C) \cup (B - C) && \text{(definition of set difference)},\end{aligned}$$

thus completing the proof. □

You can also give the appearance of two columns in the following way.

Proposition. *The identity* $(A \cup B) - C = (A - C) \cup (B - C)$ *holds for all sets* A*,* B*, and* C*.*

Proof. Observe,

$$\begin{aligned}
& (A \cup B) - C \\
=& (A \cup B) \cap C^c && \text{(definition of set difference)} \\
=& (A \cap C^c) \cup (B \cap C^c) && \text{(distributive law)} \\
=& (A - C) \cup (B - C) && \text{(definition of set difference),}
\end{aligned}$$

thus completing the proof. □