Generating Functions and the Fibonacci Sequence

Sarah Oligmueller

Department of Mathematics
Nebraska Wesleyan University

June 14, 2015

Fibonacci Sequence

Fun Fact: November 23rd is Fibonacci Day!
(1, 1, 2, 3)

Definition

The Fibonacci sequence is defined by the recurrence relation $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$.
So, the sequence is 1, 1, 2, 3, 5, 8, 13...

What if?

- We want to know the 10th number of the sequence?
- We want to know the 20th or 50th term?

Fibonacci Sequence

What if?

- We want to know the 10th number of the sequence?
- We want to know the 20th or 50th term?

Generating Functions

Definition

Let $\{f_n\}_{n\geq 0}$ be a sequence of real numbers. Then, the formal power series

$$F(x) = \sum_{n\geq 0} f_n x^n$$

is called the ordinary generating function of the sequence $\{f_n\}_{n\geq 0}$.

Example

We have invested 1000 dollars into a savings account that pays five percent interest at the end of each year. At the beginning of each year, we deposit another 500 dollars into this account. How much money will be in this account after n years?

The recurrence relation is $a_{n+1} = 1.05a_n + 500$ with $a_0 = 1000$.

Example

We have invested 1000 dollars into a savings account that pays five percent interest at the end of each year. At the beginning of each year, we deposit another 500 dollars into this account. How much money will be in this account after n years?

The recurrence relation is $a_{n+1} = 1.05a_n + 500$ with $a_0 = 1000$.

Example

The recurrence relation is $a_{n+1} = 1.05a_n + 500$ with $a_0 = 1000$. Let

$$G(x) = \sum_{n \geq 0} a_n x^n$$

be the generating function of the sequence $\{a_n\}_{n \geq 0}$.

Now, multiply both sides of the recurrence relation by x^{n+1} and sum over all non-negative integers n. We get

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = \sum_{n \geq 0} 1.05 a_n x^{n+1} + \sum_{n \geq 0} 500 x^{n+1}.$$

Note:

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = G(x) - a_0 \text{ and } \sum_{n \geq 0} 1.05 a_n x^{n+1} = 1.05 x G(x).$$

Example

The recurrence relation is $a_{n+1} = 1.05a_n + 500$ with $a_0 = 1000$. Let

$$G(x) = \sum_{n \geq 0} a_n x^n$$

be the generating function of the sequence $\{a_n\}_{n \geq 0}$.

Now, multiply both sides of the recurrence relation by x^{n+1} and sum over all non-negative integers n. We get

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = \sum_{n \geq 0} 1.05 a_n x^{n+1} + \sum_{n \geq 0} 500 x^{n+1}.$$

Note:

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = G(x) - a_0 \text{ and } \sum_{n \geq 0} 1.05 a_n x^{n+1} = 1.05xG(x).$$

Example

The recurrence relation is $a_{n+1} = 1.05a_n + 500$ with $a_0 = 1000$. Let

$$G(x) = \sum_{n\geq 0} a_n x^n$$

be the generating function of the sequence $\{a_n\}_{n\geq 0}$.

Now, multiply both sides of the recurrence relation by x^{n+1} and sum over all non-negative integers n. We get

$$\sum_{n\geq 0} a_{n+1}x^{n+1} = \sum_{n\geq 0} 1.05a_n x^{n+1} + \sum_{n\geq 0} 500x^{n+1}.$$

Note:

$$\sum_{n\geq 0} a_{n+1}x^{n+1} = G(x) - a_0 \text{ and } \sum_{n\geq 0} 1.05a_n x^{n+1} = 1.05xG(x).$$

Example

Reminder: The Taylor Series for $\frac{1}{1-x}$ is $\sum_{n\geq 0} x^n$. So,

$$\sum_{n\geq 0} 500x^{n+1} = \frac{500x}{1-x}.$$

By substitution, we have

$$G(x) - a_0 = 1.05xG(x) + \frac{500x}{1-x}.$$

Solving for G(x), we obtain

$$G(x) = \frac{1000}{1-1.05x} + \frac{500x}{(1-x)(1.105x)}.$$

By partial fractions,

$$\frac{500x}{(1-x)(1.105x)} = 10000\left(\frac{1}{1-1.05x} - \frac{1}{1-x}\right).$$

Example

Reminder: The Taylor Series for $\frac{1}{1-x}$ is $\sum_{n\geq 0} x^n$. So,

$$\sum_{n\geq 0} 500x^{n+1} = \frac{500x}{1-x}.$$

By substitution, we have

$$G(x) - a_0 = 1.05xG(x) + \frac{500x}{1-x}.$$

Solving for G(x), we obtain

$$G(x) = \frac{1000}{1-1.05x} + \frac{500x}{(1-x)(1.105x)}.$$

By partial fractions,

$$\frac{500x}{(1-x)(1.105x)} = 10000\left(\frac{1}{1-1.05x} - \frac{1}{1-x}\right).$$

Example

Reminder: The Taylor Series for $\frac{1}{1-x}$ is $\sum_{n\geq 0} x^n$. So,

$$\sum_{n\geq 0} 500x^{n+1} = \frac{500x}{1-x}.$$

By substitution, we have

$$G(x) - a_0 = 1.05xG(x) + \frac{500x}{1-x}.$$

Solving for G(x), we obtain

$$G(x) = \frac{1000}{1-1.05x} + \frac{500x}{(1-x)(1.105x)}.$$

By partial fractions,

$$\frac{500x}{(1-x)(1.105x)} = 10000\left(\frac{1}{1-1.05x} - \frac{1}{1-x}\right).$$

Example

Reminder: The Taylor Series for $\frac{1}{1-x}$ is $\sum_{n\geq 0} x^n$. So,

$$\sum_{n\geq 0} 500x^{n+1} = \frac{500x}{1-x}.$$

By substitution, we have

$$G(x) - a_0 = 1.05xG(x) + \frac{500x}{1-x}.$$

Solving for G(x), we obtain

$$G(x) = \frac{1000}{1-1.05x} + \frac{500x}{(1-x)(1.105x)}.$$

By partial fractions,

$$\frac{500x}{(1-x)(1.105x)} = 10000\left(\frac{1}{1-1.05x} - \frac{1}{1-x}\right).$$

Example

Replacing with Taylor series, we obtain

$$G(x) = 1000 \sum_{n \geq 0} (1.05x)^n + 10000 \sum_{n \geq 0} (1.05^n - 1)x^n.$$

This means,

$$a_n = 1000 * 1.05^n + 10000(1.05^n - 1)$$
$$= 11000 * 1.05^n - 10000.$$

This how much money will be in the account after n years!

Example

Replacing with Taylor series, we obtain

$$G(x) = 1000 \sum_{n \geq 0} (1.05x)^n + 10000 \sum_{n \geq 0} (1.05^n - 1)x^n.$$

This means,

$$\begin{aligned} a_n &= 1000 * 1.05^n + 10000(1.05^n - 1) \\ &= 11000 * 1.05^n - 10000. \end{aligned}$$

This how much money will be in the account after n years!

Example

Replacing with Taylor series, we obtain

$$G(x) = 1000 \sum_{n \geq 0} (1.05x)^n + 10000 \sum_{n \geq 0} (1.05^n - 1)x^n.$$

This means,

$$\begin{aligned} a_n &= 1000 * 1.05^n + 10000(1.05^n - 1) \\ &= 11000 * 1.05^n - 10000. \end{aligned}$$

This how much money will be in the account after n years!

Fibonacci Example

The recurrence relation $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$. Let

$$F(x) = \sum_{n \geq 0} f(n)x^n$$

be the generating function.
Multiply both sides of the recurrence relation by x^{n+2} and sum it over all non-negative integers n. We get

$$\sum_{n \geq 0} f(n+2)x^{n+2} = \sum_{n \geq 0} f(n+1)x^{n+2} + \sum_{n \geq 0} f(n)x^{n+2}.$$

Using the defined generating function, we obtain

$$F(x) - x - 1 = x(F(x) - 1) + x^2 F(x).$$

Fibonacci Example

The recurrence relation $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$. Let

$$F(x) = \sum_{n \geq 0} f(n)x^n$$

be the generating function.
Multiply both sides of the recurrence relation by x^{n+2} and sum it over all non-negative integers n. We get

$$\sum_{n \geq 0} f(n+2)x^{n+2} = \sum_{n \geq 0} f(n+1)x^{n+2} + \sum_{n \geq 0} f(n)x^{n+2}.$$

Using the defined generating function, we obtain

$$F(x) - x - 1 = x(F(x) - 1) + x^2 F(x).$$

Fibonacci Example

The recurrence relation $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$.
Let

$$F(x) = \sum_{n \geq 0} f(n)x^n$$

be the generating function.
Multiply both sides of the recurrence relation by x^{n+2} and sum it over all non-negative integers n. We get

$$\sum_{n \geq 0} f(n+2)x^{n+2} = \sum_{n \geq 0} f(n+1)x^{n+2} + \sum_{n \geq 0} f(n)x^{n+2}.$$

Using the defined generating function, we obtain

$$F(x) - x - 1 = x(F(x) - 1) + x^2 F(x).$$

Fibonacci Example

Solving for F(x), we obtain

$$F(x) = \frac{1}{1 - x - x^2}.$$

Note that the two roots of $1 - x - x^2$ are

$$r_1 = -\frac{1 + \sqrt{5}}{2} \text{ and } r_2 = -\frac{1 - \sqrt{5}}{2}.$$

By partial fractions, we get

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{x - r_2} - \frac{1}{x - r_1} \right).$$

Fibonacci Example

Solving for F(x), we obtain

$$F(x) = \frac{1}{1 - x - x^2}.$$

Note that the two roots of $1 - x - x^2$ are

$$r_1 = -\frac{1+\sqrt{5}}{2} \text{ and } r_2 = -\frac{1-\sqrt{5}}{2}.$$

By partial fractions, we get

$$F(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{x - r_2} - \frac{1}{x - r_1}\right).$$

Fibonacci Example

Now,

$$\begin{aligned}
F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{x-r_2}-\frac{1}{x-r_1}\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{r_1}{r_1}\cdot\frac{1}{x-r_2}-\frac{r_2}{r_2}\cdot\frac{1}{x-r_1}\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{r_1}{r_1x-1}-\frac{r_2}{r_2x-1}\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{r_2}{1-r_2x}-\frac{r_1}{1-r_1x}\right)\\
&= \frac{1}{\sqrt{5}}\left(r_2\sum_{n\geq 0} r_2^n x^n - r_1\sum_{n\geq 0} r_1^n x^n\right).
\end{aligned}$$

Fibonacci Example

Now,

$$
\begin{aligned}
F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{x-r_2} - \frac{1}{x-r_1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{r_1}{r_1}\cdot\frac{1}{x-r_2} - \frac{r_2}{r_2}\cdot\frac{1}{x-r_1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{r_1}{r_1 x-1} - \frac{r_2}{r_2 x-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{r_2}{1-r_2 x} - \frac{r_1}{1-r_1 x}\right) \\
&= \frac{1}{\sqrt{5}}\left(r_2\sum_{n\geq 0} r_2^n x^n - r_1\sum_{n\geq 0} r_1^n x^n\right).
\end{aligned}
$$

Fibonacci Example

Now,

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{x-r_2} - \frac{1}{x-r_1}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{r_1}{r_1}\cdot\frac{1}{x-r_2} - \frac{r_2}{r_2}\cdot\frac{1}{x-r_1}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{r_1}{r_1x-1} - \frac{r_2}{r_2x-1}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{r_2}{1-r_2x} - \frac{r_1}{1-r_1x}\right) \\ &= \frac{1}{\sqrt{5}}\left(r_2\sum_{n\geq 0} r_2^n x^n - r_1\sum_{n\geq 0} r_1^n x^n\right). \end{aligned}$$

Fibonacci Example

Now,

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{x-r_2} - \frac{1}{x-r_1}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{r_1}{r_1}\cdot\frac{1}{x-r_2} - \frac{r_2}{r_2}\cdot\frac{1}{x-r_1}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{r_1}{r_1 x-1} - \frac{r_2}{r_2 x-1}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{r_2}{1-r_2 x} - \frac{r_1}{1-r_1 x}\right) \\ &= \frac{1}{\sqrt{5}}\left(r_2\sum_{n\geq 0} r_2^n x^n - r_1\sum_{n\geq 0} r_1^n x^n\right). \end{aligned}$$

Fibonacci Example

So,

$$F(x) = \frac{1}{\sqrt{5}} \left(r_2 \sum_{n \geq 0} r_2^n x^n - r_1 \sum_{n \geq 0} r_1^n x^n \right).$$

Thus,

$$f_n = \frac{1}{\sqrt{5}} \left(r_2^{n+1} - r_1^{n+1} \right).$$

By substituting r_1 and r_2 into the equation, we get

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

Note: this equation is set up so $f_n = F_{n+1}$ since the series indices begin at 0, but the Fibonacci indices begin at 1.

Fibonacci Example

So,

$$F(x) = \frac{1}{\sqrt{5}} \left(r_2 \sum_{n \geq 0} r_2^n x^n - r_1 \sum_{n \geq 0} r_1^n x^n \right).$$

Thus,

$$f_n = \frac{1}{\sqrt{5}} \left(r_2^{n+1} - r_1^{n+1} \right).$$

By substituting r_1 and r_2 into the equation, we get

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

Note: this equation is set up so $f_n = F_{n+1}$ since the series indices begin at 0, but the Fibonacci indices begin at 1.

Fibonacci Example

So,

$$F(x) = \frac{1}{\sqrt{5}} \left(r_2 \sum_{n \geq 0} r_2^n x^n - r_1 \sum_{n \geq 0} r_1^n x^n \right).$$

Thus,

$$f_n = \frac{1}{\sqrt{5}} \left(r_2^{n+1} - r_1^{n+1} \right).$$

By substituting r_1 and r_2 into the equation, we get

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

Note: this equation is set up so $f_n = F_{n+1}$ since the series indices begin at 0, but the Fibonacci indices begin at 1.

Does it work?

$$F_{10} = f_9 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{10} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{10} = 55$$

If we go through the sequence,

1, 1, 2, 3, 5, 8, 13, 21, 34, 55

55 is the 10th term!

It works!

It works!

$$F_{20} = f_{19} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{20} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{20} = 6,765$$

$$F_{50} = f_{49} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{50} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{50} = 12,586,269,025$$

Take a Closer Look

With our equation,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

the second term goes to zero quickly. So, we can remove that term

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

and still get fairly accurate results.

Take a Closer Look

With our equation,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

the second term goes to zero quickly. So, we can remove that term

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

and still get fairly accurate results.

Take a Closer Look

$$F_{10} = f_9 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{10} = 55.00368$$

$$F_{20} = f_{19} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{20} = 6,765.00003$$

$$F_{50} = f_{49} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{50} = 12,586,269,025.00000$$

Thanks!

Thanks for listening!

Search the web for “Generating Functions” to learn more.