Generating Functions and the Fibonacci Sequence

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Introduction

The Fibonacci sequence is a well known sequence in mathematics developed by adding the two previous terms to get the next term. Defined in the 13th century by an Italian mathematician, Leonardo Fibonacci, the recurrence relation for the Fibonacci sequence is $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$. So, the sequence would be 1, 1, 2, 3, 5, 8, 13...

What if we wanted to find the 10th term of the Fibonacci sequence? This would not be a very difficult task, right? We could walk through the sequence and find the 10th term fairly easily with minimal effort. However, what if we want to find the 20th term or the 50th term? This task would consume much more time and be very monotonous to complete.

Wouldn't it be nice if there was an easier way to find these terms? Luckily, there is a solution to this problem! We can find a rational function whose power series has precisely the Fibonacci numbers as coefficients. This expression will allow us to calculate any Fibonacci number without going through the recurrence relation.

Ordinary Generating Functions

Ordinary generating functions are used in mathematics to condense infinite sequences into a single expression. The expression found will allow for any term to be calculated without using the recurrence relation.

Definition. *Let* $\{f_n\}_{n\geq 0}$ *be a sequence of real numbers. The formal power series*

$$F(x) = \sum_{n\geq 0} f_n x^n$$

is called the ordinary generating function of the sequence $\{f_n\}_{n\geq 0}$.

Taylor series from calculus also come into play when we are finding the expression by allowing us to manipulate the expression as necessary.

We will use an example to further demonstrate an ordinary generating function. Before we find the ordinary generating function for the Fibonacci sequence, we will find a different ordinary generating function with a recurrence relation that contains one previous term rather than two.

Example

We have invested 1000 dollars into a savings account that pays five percent interest at the end of each year. At the beginning of each year, we deposit another 500 dollars into this account. How much money will be in this account after n years?

Let a_n be the account balance after n years. Since we initially invest 1000 dollars into the account, $a_0 = 1000$. The five percent interest at the end of each year will be calculated by multiplying 1.05 by number of previous years. The 500 dollars we add into the account every year will be calculated by adding 500 dollars. So, the recurrence relation is $a_{n+1} = 1.05a_n + 500$ with $a_0 = 1000$. Let

$$G(x) = \sum_{n\geq 0} a_n x^n$$

be the formal power series of the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$. We will use the recurrence relation to find the coefficients for the generating function. First, multiply both sides of the recurrence relation by x^{n+1} and sum over all non-negative integers n. We get

$$\sum_{n\geq 0} a_{n+1}x^{n+1} = \sum_{n\geq 0} 1.05a_n x^{n+1} + \sum_{n\geq 0} 500x^{n+1}.$$

Note that

$$\sum_{n\geq 0} a_{n+1}x^{n+1} = G(x) - a_0 \text{ and } \sum_{n\geq 0} 1.05a_n x^{n+1} = 1.05xG(x)$$

based on our defined power series $G(x)$. If we think back to calculus, the Taylor series for $\frac{1}{1-x}$ is $\sum_{n\geq 0} x^n$. So,

$$\sum_{n\geq 0} 500x^{n+1} = \frac{500x}{1-x}.$$

By substituting these terms into our equation, we have

$$G(x) - 1000 = 1.05xG(x) + \frac{500x}{1-x}.$$

Solving for $G(x)$, we obtain

$$G(x) = \frac{1000}{1-1.05x} + \frac{500x}{(1-x)(1.105x)}.$$

By partial fractions,

$$\frac{500x}{(1-x)(1.105x)} = 10000\left(\frac{1}{1-1.05x} - \frac{1}{1-x}\right).$$

Substituting in the partial fractions, we see that

$$G(x) = \frac{1000}{1-1.05x} + 10000\left(\frac{1}{1-1.05x} - \frac{1}{1-x}\right).$$

Replacing with Taylor series, we have

$$G(x) = 1000\sum_{n\geq 0}(1.05x)^n + 10000\sum_{n\geq 0}(1.05^n - 1)x^n.$$

Finally, we obtain our answer which is

$$\begin{aligned} a_n &= 1000 * 1.05^n + 10000(1.05^n - 1) \\ &= 11000 * 1.05^n - 10000. \end{aligned}$$

This how much money will be in the account after n years!

Fibonacci Example

Now that we have learned how to find an ordinary generating function for a recurrence relation using one previous term, we will now find an ordinary generating function for the Fibonacci sequence which uses two previous terms. The recurrence relation for the Fibonacci sequence is $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$. Let

$$F(x) = \sum_{n\geq 0} f_n x^n$$

be the ordinary generating function for the Fibonacci sequence. Now, we will multiply both sides of the recurrence relation by x^{n+2} and sum it over all non-negative integers n. We get

$$\sum_{n\geq 0} f_{n+2}x^{n+2} = \sum_{n\geq 0} f_{n+1}x^{n+2} + \sum_{n\geq 0} f_n x^{n+2}.$$

Notice that we multiply by x^{n+2} instead of x^{n+1} like in the previous example since the recurrence relation for the Fibonacci sequence uses two previous terms whereas our other example used one previous term.

Using the defined formal power series, we see that

$$F(x) - x - 1 = x(F(x) - 1) + x^2 F(x).$$

Solving for F(x), we obtain

$$F(x) = \frac{1}{1 - x - x^2}.$$

Note that the two roots of $1 - x - x^2$ are

$$r_1 = -\frac{1+\sqrt{5}}{2} \text{ and } r_2 = -\frac{1-\sqrt{5}}{2}$$

which are inverses of each other meaning $r_1 \cdot r_2 = 1$. We will use this fact later in developing our expression.

By partial fractions, we get

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{x - r_2} - \frac{1}{x - r_1} \right).$$

Now,

$$\begin{aligned}
F(x) &= \frac{1}{\sqrt{5}} \left(\frac{1}{x - r_2} - \frac{1}{x - r_1} \right) \\
&= \frac{1}{\sqrt{5}} \left(\frac{r_1}{r_1} \cdot \frac{1}{x - r_2} - \frac{r_2}{r_2} \cdot \frac{1}{x - r_1} \right) \\
&= \frac{1}{\sqrt{5}} \left(\frac{r_1}{r_1 x - 1} - \frac{r_2}{r_2 x - 1} \right) && \text{since } r_1 \cdot r_2 = 1 \\
&= \frac{1}{\sqrt{5}} \left(\frac{r_2}{1 - r_2 x} - \frac{r_1}{1 - r_1 x} \right) \\
&= \frac{1}{\sqrt{5}} \left(r_2 \sum_{n \geq 0} r_2^n x^n - r_1 \sum_{n \geq 0} r_1^n x^n \right).
\end{aligned}$$

By equating coefficients,

$$f_n = \frac{1}{\sqrt{5}} \left(r_2^{n+1} - r_1^{n+1} \right).$$

By substituting r_1 and r_2 into the equation, we get

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

Notice that this equation is set up so $f_n = F_{n+1}$ since the series indices begin at 0, but the Fibonacci indices begin at 1. For example,

$$F_{10} = f_9 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{10} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{10} = 55.$$

If we go through the sequence,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55...$$

we discover that 55 is the 10th term! Now, let's find the 20th term and the 50th term of the Fibonacci sequence as we initially inquired. So,

$$F_{20} = f_{19} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{20} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{20} = 6,765$$

and

$$F_{50} = f_{49} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{50} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{50} = 12,586,269,025.$$

Wasn't using the equation much easier than going through the entire sequence?

Take a Closer Look

If we take a closer look at our equation,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

we find that the second term goes to zero quickly since the term is less than 1. So, we can actually just remove that term

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

and still get accurate results rounding to the nearest integer. This is true for any number since the term $\frac{1}{\sqrt{5}}r_1^{n+1}$ is already less than 0.5 for $n = 0$ and is strictly decreasing in n. This means that $\frac{1}{\sqrt{5}}r_1^{n+1}$ is less than 0.5 for all n which is why rounding to the nearest integer will always give you the correct Fibonacci number. Using the 10th Fibonacci number again as a simple example,

$$F_{10} = f_9 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{10} = 55.00368.$$

So, even the 10th term of the of the expression is very close to the actual value of the 10th Fibonacci sequence. As stated before, we would get the correct Fibonacci number by rounding to the nearest integer. Calculating the 20th term and 50th term again, we see that

$$F_{20} = f_{19} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{20} = 6,765.00003$$

and

$$F_{50} = f_{49} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{50} = 12,586,269,025.00000.$$

So, we get even more accurate results before rounding!

Conclusion

We have learned how to find the ordinary generating function by using the recursion relation to discover the coefficients for the formal power series. By utilizing Taylor series, we manipulate the power series until we find a single expression for the infinite sequence.

Ordinary generating functions are useful in mathematics by allowing us to condense infinite sequences into a single expression that computes each term in the sequence without directly using the recursion relation. With this technique, we are able to calculate the 20th term and the 50th term of the Fibonacci sequence without going through the sequence with brute force. This time saving expression allows us to further understand the infinite sequences such as the Fibonacci sequence and calculate terms more quickly.