The Matrix: Expressing Trees

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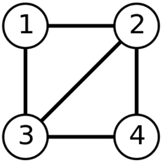
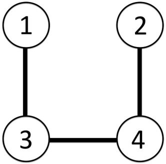
Trees

Theorem

Let G be a connected simple graph on n vertices. Then the following are equivalent:
(1) G is minimally connected, that is, if we remove any edge of G, then the obtained graph G' will not be connected.
(2) G does not contain a cycle.
(3) For each pair of vertices (x,y), there is exactly one path joining x and y.



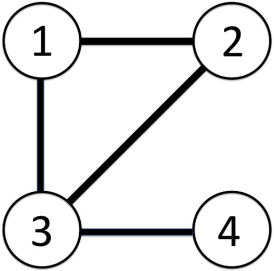
not minimally connected



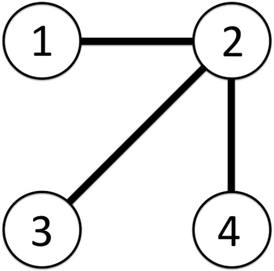
minimally connected

Trees

A connected simple graph G satisfying any criteria of the previous theorem is called a **tree**.



not a tree

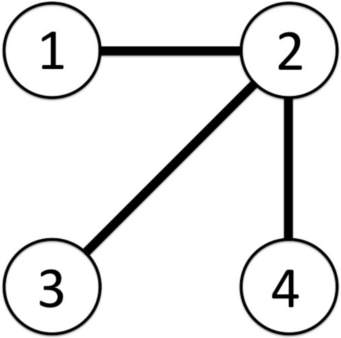


tree

Trees

Theorem

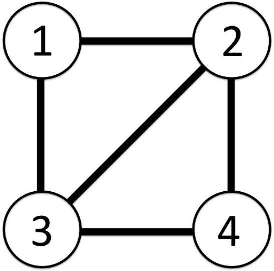
All trees on n vertices have n-1 edges. Conversely, all connected graphs on n vertices with exactly n-1 edges are trees.



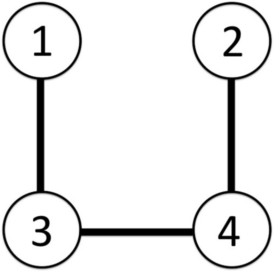
A tree on $n = 4$ vertices has 3 edges.

Spanning Trees

If G is a connected graph, we say that T is a **spanning tree** of G if G and T have the same vertex set, and each edge of T is also an edge of G.



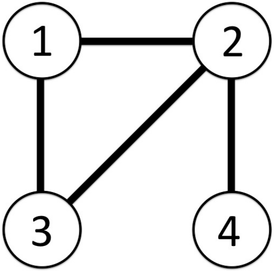
graph G



T = spanning tree of G

The Adjacency Matrix

Let G be an undirected graph on n labeled vertices, and define an $n \times n$ matrix $A = A_G$ by setting $A_{i,j}$ equal to the number of edges between vertices i and j. Then A is called the **adjacency matrix** of G.



graph G

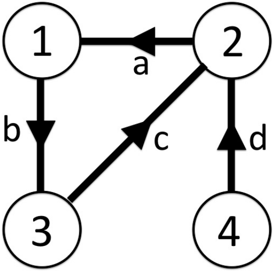
$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

adjacency matrix A

The Incidence Matrix

Let G be a directed graph without loops. Let $\{v_1, v_2, \ldots, v_n\}$ denote the vertices of G, and let $\{e_1, e_2, \ldots, e_m\}$ denote the edges of G. Then the **incidence matrix** of G is the $n \times m$ matrix A defined by

- $a_{i,j} = 1$ if v_i is the head of e_j,
- $a_{i,j} = -1$ if v_i is the tail of e_j, and
- $a_{i,j} = 0$ otherwise.



graph G

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

incidence matrix

Computing the Number of Spanning Trees

Theorem

Let G be a directed graph without loops, and let A be the incidence matrix of G. Remove any row from A, and let A_0 be the remaining matrix. Then the number of spanning trees of G is $det A_0 A_0^T$.

Proof.

Assume, without loss of generality, that the last row of A is removed. Let B be an $(n-1) \times (n-1)$ submatrix of A_0. We will show that $|\det B| = 1$ if and only if the subgraph G' corresponding to the columns of B is a spanning tree, and $\det B = 0$ otherwise using cases.

(1) Assume there is a vertex v_i $(i \neq n)$ of degree one in G'. Then, if G' is a spanning tree of G, $G' - v_i$ must also be a spanning tree of $G - v_i$. So $|\det B| = 1$.

(2) Assume that G' has no vertices of degree one.

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Computing the Number of Spanning Trees

Proof (Cont.)

Then G' is not a spanning tree. Since G' has $n-1$ edges and is not a spanning tree, there must be a vertex of degree zero. If this vertex is not v_n, then B has a zero row, and $\det B = 0$. If this vertex is v_n, then the rows of B are linearly dependent, and $\det B = 0$.

So $|\det B| = 1$ if G' is a spanning tree of G and $\det B = 0$ otherwise. By the Binet-Cauchy formula, $\det A_0 A_0^T = \sum (\det B)^2$ where the sum ranges over all $(n-1) \times (n-1)$ submatrices B of A_0. Thus the number of spanning trees of G is $\det A_0 A_0^T$. $\square$

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Computing the Number of Spanning Trees

Theorem (Matrix Tree Theorem)

Let U be a simple undirected graph. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of U. Define the $(n-1) \times (n-1)$ matrix L_0 by

$$l_{i,j} = \begin{cases} \text{the degree of } v_i \text{ if } i = j \\ -1 \text{ if } i \neq j, \text{and } v_i \text{ and } v_j \text{ are connected, and} \\ 0 \text{ otherwise,} \end{cases}$$

where $1 \leq i, j \leq n-1$. Then U has exactly $detL_0$ spanning trees.

Consider converting U into a directed graph G by replacing each edge in U with a pair of directed edges, one going in each direction. One can show that $A_0 A_0^T = 2L_0$. Then $\det(A_0 A_0^T) = 2^{n-1}\det L_0$.

Spanning Trees of K_n

Proposition

The number of spanning trees of K_n is n^{n-2}.

Proof.

By the Matrix-Tree Theorem, the matrix L_0 associated with K_n is as follows:

$$\begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \dots & & & \\ -1 & -1 & \dots & n-1 \end{pmatrix}$$

Spanning Trees of K_n

Proof (Cont.)

After row reducing to a triangular matrix, we get

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & n & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & n \end{pmatrix}$$

Thus $\det L_0 = n^{n-2}$. □

Note that this is a proof of Cayley's Theorem which states that for any positive integer n, the number of all trees with vertex set $[n]$ is $A_n = n^{n-2}$.

Complete Bipartite Graph

Let A be a set of m vertices, and let B be a set of n vertices. Connect each vertex of A to each vertex of B by an edge. Denote this graph by $K_{m,n}$. The graph $K_{m,n}$ is called a **complete bipartite graph**. Note there is no edge within A or within B in $K_{m,n}$. Find the number of spanning trees of $K_{m,n}$.

Solution: By the Matrix-Tree Theorem, the matrix L_0 associated with $K_{m,n}$ is of the following form:

$$\begin{pmatrix} n & \dots & 0 & -1 & \dots & -1 \\ \dots & & & & & \\ 0 & \dots & n & -1 & \dots & -1 \\ -1 & \dots & -1 & m & \dots & 0 \\ \dots & & & & & \\ -1 & \dots & -1 & 0 & \dots & m \end{pmatrix}$$

Complete Bipartite Graph

After some arranging of the rows, we get

$$\begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ \dots \\ 0 & \dots & n & -1 & \dots & -1 \\ 0 & \dots & 0 & m & \dots & 0 \\ \dots \\ 0 & \dots & 0 & 0 & \dots & m \end{pmatrix}$$

Now, $\det L_0 = n^{m-1}m^{n-1}$. So there are $n^{m-1}m^{n-1}$ spanning trees of $K_{m,n}$.

Thank You

Thank you for listening!