Ramsey Theorem

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Abstract

Ramsey's theorem states that parties will always contain a large cluster of friends or a large cluster of strangers provided you invite enough people. We will look at some examples of small parties and move on to discover that Ramsey's theorem applies even to a party with infinitely many guests.

1 An Introduction

We will begin our discussion of Ramsey Theory by a well-known introductory problem. Imagine sitting at a bar in a foreign country. A momentary distraction finds one of our friends leaving with a woman of the night and the other disappears with questionable characters. Realizing we are now alone with no way of getting home, it seems only natural to distract our minds with a mathematical dilemma. Gazing through the crowd we notice a group of six people talking, some of whom seem to be more at ease, while others seem less comfortable. We can prove that either three of them know each other or three of them do not know each other.

Allow each person to represent a vertex on a complete graph. Color the edge between two distinct vertices blue if the corresponding people know each other, and red if they do not. As this procedure is continued, our claim will be proven if we can show a monochromatic triangle exists in every configuration of 15 colored edges.

Choose any one vertex v, and note v is of degree 5. Dictated by the Pigeon Hole Principle, we know v must have at least 3 edges comprised of the same color. Without loss of generality assume that those edges are blue. Let x, y, and z be the receiving vertices of the blue edges. Notice that if any one of the edges linking the vertices x, y, or z are blue, then we have a group of friends. If the vertices x, y, and z do not contain any blue lines, then we automatically have a group of strangers. Thus, in any group of 6 bar-goers, there exists a group of three people that know each other, or three who do not know each other.

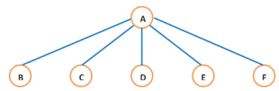


Figure 1: figure
Any blue edge creates a blue subgraph.

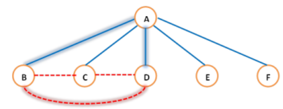
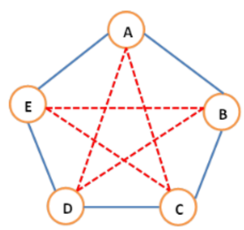


Figure 2: figure
All red edges make a red subgraph.

This phenomena of ascertaining structure within a large-enough random group is called Ramsey's Theorem. It is important to note that this result is tight. That is, it will not remain true if only five people were in the group. For example,



does not provide either a group of three friends nor three strangers. This contradiction demonstrates the rigidity of the result.

2 Ramsey Theorem for Graphs

Theorem. *Let k and l be two positive integers, both of which are at least two. Then there exists a (minimum) positive integer $R(k,l)$ so that if we color the edges of a complete graph with $R(k,l)$ vertices red and blue, then this graph will either have a K_k subgraph with only red edges, or a K_l subgraph with only blue edges.*

Proof. To prove this statement, we will utilize induction. First, we must prove the base case; that is to show that $R(2,k)$ and $R(l,2)$ exist for all k,l. Consider the positive integer $R(l,2)$. This is the number of vertices required in a complete graph in order to guarantee a monochromatic blue K_l subgraph, or a red K_2 subgraph. Take a complete graph on l vertices. Either there is a red edge producing a red K_2 subgraph or not providing a blue K_l. Therefore, $R(l,2) = l$. Similarly, we have $R(2,k) = k$.

We next show

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

for all $k,l \geq 2$. Take a complete graph on $R(k,l-1) + R(k-1,l)$ vertices. Select any one vertex v. Note that v is of degree $R(k,l-1)+R(k-1,l)-1$ which exist by induction. Otherwise, the red and blue edges only sum to at most $(R(k,l-1)+R(k-1,l)-2)$. It is clear that v can have at least $R(k,l-1)$

blue edges, or a minimum of $R(k-1,l)$ red edges. Assume v has at least $R(k,l-1)$ blue edges adjacent to itself. Let B denote the $R(k,l-1)$-element set of the other endpoints of these blue edges. By definition, the set B contains a monochromatic blue K_k or a monochromatic red K_{l-1}, which can be completed to a monochromatic red K_l by adding v. In either case, we produce one of the desired subgraphs. Let's consider the case of v having $R(k-1)$ red edges. Similarly, denote r to be the $R(k-1,l)$-element set of the other endpoints of these red edges. Once again, r either contains monochromatic blue K_l or a monochromatic red K_{k-1} completed to a monochromatic K_l by adding vertex v. Either way, a desired subgraph was produced.

We have demonstrated that given $R(k,l-1)+R(k-1,l)$ vertices on a graph, then we have enough vertices to guarantee a monochromatic blue K_k or a monochromatic red K_l. Thus, the inductive step holds, and we have proven the Ramsey Theorem. □

This theorem only proves that the Ramsey Number, $R(r,s)$, always exists. It does not necessarily provide an exact integer value. The table below[?] represents a few Ramsey Numbers for values r,s.

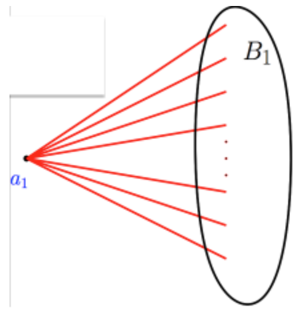
r,s	1	2	3	4	5	6	7	8	9	10
1	**1**									
2	1	**2**								
3	1	3	**6**							
4	1	4	9	**18**						
5	1	5	14	25	**43–49**					
6	1	6	18	36–41	58–87	**102–165**				
7	1	7	23	49–61	80–143	113–298	**205–540**			
8	1	8	28	58–84	101–216	132–495	217–1031	**282–1870**		
9	1	9	36	73–115	126–316	169–780	241–1713	317–3583	**565–6588**	
10	1	10	40–42	92–149	144–442	179–1171	289–2826	331–6090	581–12677	**798–23556**

As the values of r,s increase, even slightly, an exact value $R(r,s)$ is no longer provided. Why does this phenomenon occur? Consider the time it would take to create all 2-colorings on a complete graph with 43 vertices. There exist $\binom{43}{2}$ ways to select the two distinct vertices on which to create an edge. Since each edge can be colored red or blue, there are $2^{\binom{43}{2}}$ 2-colorings of a complete graph on 43 vertices. This value is approximately 6.8×10^{271}. If we were able to create one trillion graphs per second, then it would take 2.1×10^{252} years to create all 2-colorings.

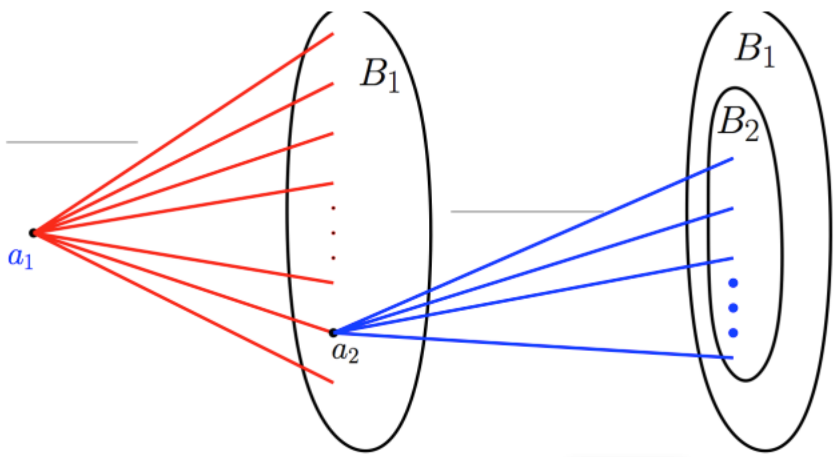
3 The Infinite Case

As stated previously, when the values of r,s increase slightly we are no longer guaranteed one positive integer $R(r,s)$. Interestingly enough, however, another result of Ramsey Theory allows us to guarantee an infinite monochromatic subgraph on an infinite number of vertices. Let us discuss graphically how we are able to prove such a result.

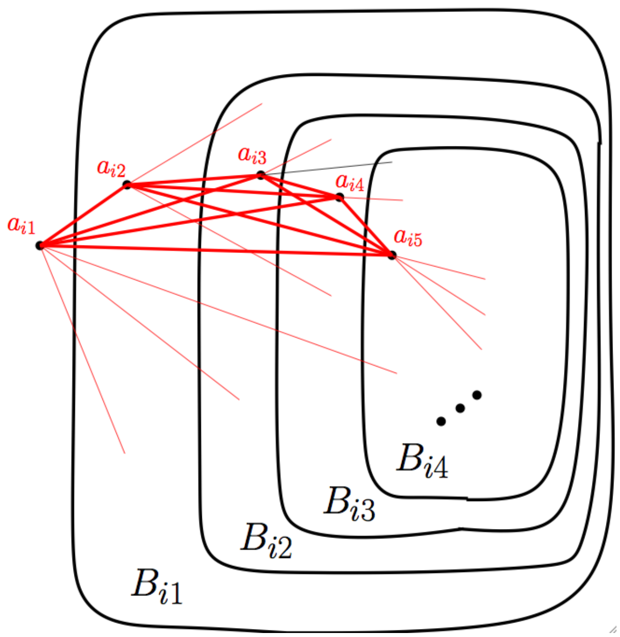
Choose any vertex a_1 in the complete graph on infinitely many vertices. Note that infinitely many edges stem from a_1. Since infinitely many edges exist with only two colors, then infinitely many of these edges must be colored with one of the colors. Without loss of generality, allow the edges to be red.



Define B_1 to be the (infinite) red neighborhood of a_1. Continuing this procedure in a similar manner, choose any $a_2 \in B_1$. Similarly, infinitely many edges emerge from a_2 to the infinite set B_2, so there are infinitely many of one color. Suppose this time they are blue.



After repeating this procedure for eternity we can analyze the outcome. We have a list of distinct vertices $a_1, a_2, a_3, \ldots,$ colored edges between all infinite sets of points, and an infinite sequence $\{B_i\}_{i=1}^{\infty}$ of nested vertex sets. Recall that each distinctive a_i has either infinitely many red or infinitely many blue edges, so there must exist a subsequence where only red or only blue edges are present. The vertices in this subsequence form the monochromatic subgraph.



Thus, we have shown that the Ramsey Theorem holds through infinity.

References

[1] Miklós Bóna. *A Walk Through Combinatorics*, second ed., World Scientific Publishing Co., 2006.

[2] Adam Azzam, Jay Cummings, Z. Norwood. *Ramsey Tehory Part 2; RT on Infinite, Colorful Graphs*, Whatever Suits Your Boat.