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Please write *only* your name on the test sheet.
Place all work and answers on the blank sheets provided.
Only attempt problems that you have not previously mastered.

13. Find the Taylor series for $\frac{1}{x}$ centered at 1.

Solution: Taking a few derivatives of $f(x) = \frac{1}{x}$ will help reveal the general form for $f^{(n)}(1)$.

n	$f^{(n)}(x)$
0	$\frac{1}{x}$
1	$\frac{-1}{x^2}$
2	$\frac{2\cdot 1}{x^3}$
3	$\frac{-3\cdot 2\cdot 1}{x^4}$
4	$\frac{4\cdot 3\cdot 2\cdot 1}{x^5}$

It appears that $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$, so $f^{(n)}(1) = (-1)^n n!$.

Substituting this into the definition of the Taylor series, we obtain

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n = \sum_{n=0}^{\infty}(1-x)^n.$$

14. Find the Maclaurin series for each of the following. (You may begin by modifying any of the series listed in the formula sheet.)
   (a) $e^{-x}\cos(2x)$ (first three nonzero terms only)
   (b) $\displaystyle\int \frac{1}{8+x^3}\,dx$ (all terms)

Solution: For part (a), we start by modifying known Maclaurin series to obtain $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ and $\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$. Since we only want the first three terms of the product $e^{-x}\cos(2x)$, we can truncate to the relevant terms of each to obtain

$$e^{-x}\cos(2x) \approx \left(1 - x + \frac{1}{2}x^2\right)\left(1-2x^2\right) \approx 1 - x - \frac{3}{2}x^2.$$

For part (b), begin with

$$\frac{1}{8+x^3} = \frac{1}{8}\cdot\frac{1}{1-\left(\frac{-x^3}{8}\right)}.$$

Modifying the Maclaurin series for $\frac{1}{1-x}$ gives

$$\frac{1}{8+x^3} = \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{-x^3}{8}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{8^{n+1}}x^{3n}.$$

With the series in hand, we can evaluate the integral

$$\int \frac{1}{8+x^3}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{8^{n+1}} x^{3n}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)8^{n+1}} x^{3n+1}.$$

15. Suppose you approximate the function e^{3x} using only the first three terms of its Maclaurin series. Give a reasonable upper bound on the error involved in using this approximation on the interval $[-1, 1]$.

Solution: If we use only three terms (that is, up to $n = 2$) of this series, then we seek to bound $|R_2(x)|$. Applying Taylor's inequality, we have

$$\begin{aligned} |R_2(x)| &\leq \frac{\left|f^{(3)}(x)\right|}{3!}|x|^3 \\ &= \frac{|27e^{3x}|}{6}|x|^3 \\ &= \frac{9}{2}\left|e^{3x}\right| |x|^3. \end{aligned}$$

On the domain $[-1, 1]$, both $|e^{3x}|$ and $|x|^3$ are maximized at $x = 1$. We can therefore conclude

$$|R_2(x)| \leq \frac{9}{2}e^3.$$

16. Provide the requested equations.
    (a) Find the vector and parametric equations of the line through the points $(-8, 1, 4)$ and $(3, -2, 4)$.
    (b) Find the vector and linear equations of the plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$.

Solution: For part (a), we start by building the vector equation $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. We may take to be any point on the line, so $\mathbf{r}_0 = \langle -8, 1, 4\rangle$ works fine. For the direction vector $\mathbf{v}$, we need a vector parallel to the line. The line segment from $(-8, 1, 4)$ to $(3, -2, 4)$ provides such a direction. We can represent the direction as

$$\mathbf{v} = \langle 3+8, -2-1, 4-4\rangle = \langle 11, -3, 0\rangle.$$

A vector equation of the line is therefore $\mathbf{r}(t) = \langle -8, 1, 4\rangle + t\,\langle 11, -3, 0\rangle$. The parametric equation arises by carrying out the vector arithmetic and considering each component separately. In other words, write $\mathbf{r}(t) = \langle -8 + 11t, 1 - 3t, 4\rangle$, which gives the parametric equations

$$\begin{aligned} x(t) &= -8 + 11t \\ y(t) &= 1 - 3t \\ z(t) &= 4 \end{aligned}$$

For part (b), we start by building the vector equation $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$. As before, we may take any point to define $\mathbf{r}_0$. Choosing $\mathbf{r}_0 = \langle 0, 1, 1\rangle$ will work. To determine a normal vector, we can use the technique from part (a) to form two vectors "in the plane" (that is, two vectors that are parallel to line segments in the plane). Labeling the points A, B, and C, respectively, we can form the vector $\mathbf{AB} = \langle 1 - 0, 0 - 1, 1 - 1\rangle = \langle 1, -1, 0\rangle$ and the vector $\mathbf{AC} = \langle 1 - 0, 1 - 1, 0 - 1\rangle = \langle 1, 0, -1\rangle$. The cross product of these two vectors will be normal to the plane, so we take

$$\mathbf{n} = \mathbf{AB} \times \mathbf{AC} = \langle 1, 1, 1\rangle .$$

Finally, we have the vector equation $\langle 1, 1, 1\rangle \cdot (\langle x, y, z\rangle - \langle 0, 1, 1\rangle)$. The linear equation comes from carrying out the vector arithmetic

$$\begin{aligned} 0 &= \langle 1, 1, 1\rangle \cdot (\langle x, y, z\rangle - \langle 0, 1, 1\rangle) \\ &= \langle 1, 1, 1\rangle \cdot \langle x, y - 1, z - 1\rangle \\ &= x + (y - 1) + (z - 1), \end{aligned}$$

which can be rewritten as $x + y + z = 2$.