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Please write *only* your name on the test sheet.
Place all work and answers on the blank sheets provided.
Only attempt problems that you have not previously mastered.

9. Determine whether each series converges or diverges. Clearly state any tests you use.

   (a) $\sum_{n=0}^{\infty} ne^{-n^2}$

   (b) $\sum_{n=2}^{\infty} \frac{n+1}{n^4-n}$

Solution: For part (a), notice $\int_0^\infty xe^{-x^2}\ dx = \frac{1}{2}$, so the related series is convergent by the Integral Test.

For part (b), consider the comparison of limits with $\frac{1}{n^3}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n^3}}{\frac{n+1}{n^4-n}} = \lim_{n\to\infty} \frac{n^4-n}{n^4+n} = 1.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^3}$ converges, the original series also converges by the Limit Comparison Test.

10. Determine whether each series converges absolutely, converges, or diverges. Clearly state any tests you use.

    (a) $\sum_{n=0}^{\infty} \frac{n^3}{(-4)^n}$

    (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

Solution: For part (a), consider the ratio of consecutive terms:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{(-4)^{n+1}}}{\frac{n^3}{(-4)^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{4n^3} = \frac{1}{4}.$$

The series is therefore absolutely convergent by the Ratio Test.

For part (b), first consider the sum of absolute values of the terms, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$, which is a divergent p-series. The original alternating series does converge, however, which can be established with the Alternating Series Test:

- $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$
- $\frac{1}{\sqrt[3]{n+1}} < \frac{1}{\sqrt[3]{n}}$ for all n

The original series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ therefore converges conditionally.

11. Determine the earliest partial sum whose remainder is less than 0.001. (You will not need to take more than ten terms.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

Solution: The series in part (a) is alternating, so we know $|R_n| \le b_{n+1} = \frac{1}{(n+1)^4}$. Solving $\frac{1}{(n+1)^4} < 0.001$ for n gives $n > 4.6$. We can therefore claim $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \approx -1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4}$ with remainder less than 0.001.

The series in part (b) has all positive terms, so we know $R_n \le \int_n^{\infty} \frac{1}{x^5}\,dx = \frac{1}{4n^4}$. Solving $\frac{1}{4n^4} < 0.001$ for n gives $n > 3.9$. We can therefore claim $\sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5}$ with remainder less than 0.001.

12. Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x-1)^n}{n^2}$.

Solution: Applying the Ratio Test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{3^{n+1}(x-1)^{n+1}}{(n+1)^2}}{\frac{3^n(x-1)^n}{n^2}}\right| &= \lim_{n\to\infty}\left|\frac{3(x-1)n^2}{(n+1)^2}\right| \\ &= 3|x-1|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} \\ &= 3|x-1|.\end{aligned}$$

Solving $3|x-1| < 1$ for x gives $\frac{2}{3} < x < \frac{4}{3}$. The Ratio Test guarantees convergence for $\frac{2}{3} < x < \frac{4}{3}$ and divergence for $x < \frac{2}{3}$ and $x > \frac{4}{3}$. The boundary points must be checked specifically.

When $x = \frac{2}{3}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges absolutely by the p-Series Test. When $x = \frac{4}{3}$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series.

Taking all this information together, we conclude that the interval of convergence is $\frac{2}{3} \leq x \leq \frac{4}{3}$.