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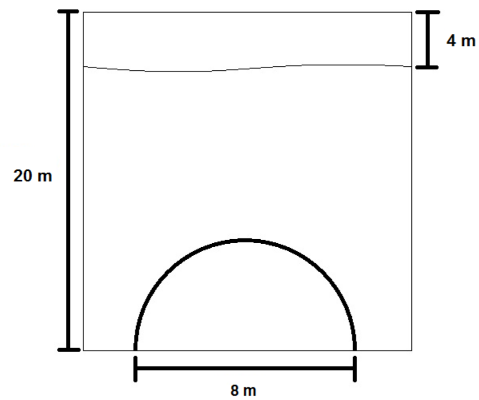
Please write *only* your name on the test sheet.
Place all work and answers on the blank sheets provided.
Only attempt problems that you have not previously mastered.

5. Set up but **do not evaluate** an integral to compute the hydrostatic force against the semicircle in the figure below.



Solution: Use a standard coordinate axis with the origin at the center of the circle. For a particular choice of y, the depth is given by $16 - y$ (check this for $y = 0$ and $y = 16$ to convince yourself). The equation of the circle (of which we only see half) is $x^2 + y^2 = 16$. Since we intend to integrate with respect to y, we express this as $x = \pm\sqrt{16 - y^2}$. A horizontal slice of the semicircle therefore has height dy and width $2\sqrt{16 - y^2}$. The values of y for which we encounter the semicircle range from 0 to 4, so we have the integral

$$19600 \int_0^4 (16 - y)\sqrt{16 - y^2}dy.$$

6. Express y **explicitly** as a function of x given the following differential equation and initial condition.

$$\frac{dy}{dx} = y^2 \sin x$$
$$y(0) = 2$$

Solution: The differential equation separates to $\frac{dy}{y^2} = \sin x dx$. Integrating both sides yields $-\frac{1}{y} = -\cos x + C$, which we can rearrange to the general solution $y(x) = \frac{1}{\cos x + C}$. Since $y(0) = 2$, we can solve $2 = \frac{1}{\cos 0 + C}$ to obtain $C = -\frac{1}{2}$. The desired particular solution is therefore $y(x) = \frac{1}{\cos x - \frac{1}{2}}$.

7. Refer to the parametric equations $x(t) = t^3 - 12t$ and $y(t) = t^2 - 1$.
   (a) Find the rectangular coordinates of all points (if any) where the tangent line is either horizontal or vertical.
   (b) Set up but **do not evaluate** an integral to find the area below the line $y = 1$ and above the curve.

Solution: For part (a), we use the fact that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ to obtain $\frac{dy}{dx} = \frac{2t}{3t^2-12}$.

The tangent line is horizontal when $\frac{dy}{dx}$ has zero numerator and nonzero denominator, which occurs only when $t = 0$. This corresponds to the rectangular coordinates $(x(0), y(0))$, which is $(0, -1)$.

The tangent line is vertical when $\frac{dy}{dx}$ has nonzero numerator and zero denominator, which occurs when $t = -2$ and $t = 2$. These correspond to the rectangular coordinates $(x(-2), y(2))$ and $(x(2), y(2))$, which are $(16, 3)$ and $(-16, 3)$, respectively.

For part (b), we first determine the bounds of integration by solving $y(t) = 1$ to find $t = -\sqrt{2}$ and $t = \sqrt{2}$. The height of a vertical slice of area is $1 - y(t)$, which is $2 - t^2$. The width of such a slice is dx, which we interpret as $\frac{dx}{dt}dt$. The width is therefore $(3t^2 - 12)dt$. These facts give the integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} (2 - t^2)(3t^2 - 12)dt$$

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8. Refer to the polar equations $r_1(\theta) = 1$ and $r_2(\theta) = 1 + \cos\theta$.
   (a) Find (using Calculus) the rectangular coordinates of all points (if any) where the line tangent to $r_1(\theta)$ is horizontal.
   (b) Set up but **do not evaluate** an integral to find the area outside $r_1(\theta)$ but inside $r_2(\theta)$.

Solution: For part (a), first parameterize x and y in terms of θ using $x(\theta) = r(\theta)\cos\theta$, and $y(\theta) = r(\theta)\sin\theta$. Doing so gives $x(\theta) = \cos\theta$ and $y(\theta) = \sin\theta$. Now that we have parametric functions, we can use the fact that $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, which results in $\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta}$. The tangent line is horizontal when $\frac{dy}{dx}$ has zero numerator and nonzero denominator, which occurs only when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. These correspond to the rectangular coordinates $\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right)$ and $\left(x\left(\frac{3\pi}{2}\right), y\left(\frac{3\pi}{2}\right)\right)$, which are $(0, 1)$ and $(0, -1)$, respectively.

For part (b), we first determine the bounds of integration by solving $r_1(\theta) = r_2(\theta)$ to find $\theta = -\frac{\pi}{2}$ and $\frac{\pi}{2}$. Recalling that the area of a sector of a circle is $\frac{1}{2}\theta r^2$, we obtain the integrals

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2}(1+\cos\theta)^2 d\theta - \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta.$$