Name: Austin Mohr
Please write *only* your name on the test sheet.
Place all work and answers on the blank sheets provided.
Only attempt problems that you have not previously mastered.

1. Evaluate each indefinite integral.

   (a) $\int 6x \sin(2x)\, dx$

   Solution: As a product of two distinct classes of functions (polynomial and trigonometric), integration by parts is a good first approach. We should choose $u = 6x$ since it becomes simpler after differentiation: $du = 6dx$. That leaves use with $dv = \sin(2x)dx$, which results in $v = -\frac{1}{2}\cos(2x)$. Using the integration by parts formula, we have

$$\begin{aligned}\int 6x \sin(2x)\, dx &= -3x\cos(2x) + 3\int \cos(2x)\, dx \\ &= -3x\cos(2x) + \frac{3}{2}\sin(2x) + C.\end{aligned}$$

   (b) $\int (1 + \cos x)^2\, dx$

   Solution: Expanding the product will help us see what we're working with.

$$\begin{aligned}\int (1+\cos x)^2\, dx &= \int \left(1 + 2\cos x + \cos^2 x\right)\, dx \\ &= \int dx + \int 2\cos x\, dx + \int \cos^2 x\, dx.\end{aligned}$$

   The first two integrals can be handled with basic techniques:

$$\int dx = x + C$$

   and

$$\int 2\cos x\, dx = 2\sin x + C.$$

   For the final integral, we can reduce the exponent with $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$.

$$\begin{aligned}\int \cos^2 x\, dx &= \frac{1}{2}\int (1 + \cos(2x))\, dx \\ &= \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C.\end{aligned}$$

Finally,

$$\begin{aligned}\int (1+\cos x)^2\, dx &= x + 2\sin x + \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C \\ &= \frac{3}{2}x + 2\sin x + \frac{1}{4}\sin(2x) + C.\end{aligned}$$

2. Evaluate each indefinite integral.

(a) $\displaystyle\int \frac{x-9}{x^2+3x-10}\, dx$

Solution: The denominator factors as $(x-2)(x+5)$, so we can use integration by partial fractions.

$$\begin{aligned}\frac{x-9}{(x-2)(x+5)} &= \frac{A}{x-2} + \frac{B}{x+5} \\ x-9 &= A(x+5) + B(x-2)\end{aligned}$$

Substituting $x = -5$ gives

$$\begin{aligned}-14 &= -7B \\ 2 &= B.\end{aligned}$$

Substituting $x = 2$ gives

$$\begin{aligned}-7 &= 7A \\ -1 &= A.\end{aligned}$$

Applying the partial fraction decomposition, we can write

$$\begin{aligned}\int \frac{x-9}{x^2+3x-10}\, dx &= -\int \frac{1}{x-2}\, dx + 2\int \frac{1}{x+5}\, dx \\ &= -\ln|x-2| + 2\ln|x+5| + C.\end{aligned}$$

(b) $\displaystyle\int x^3\sqrt{9-x^2}\, dx$

Solution: The radical in the demoninator suggests trigonometric substitution may work. Construct a triangle whose adjacent side has length x, opposite side has length $\sqrt{9-x^2}$, and hypotenuse has length 3. Using this key, we can write $\sqrt{9-x^2} = 3\sin\theta$, $x = 3\cos\theta$, and $dx = -3\sin\theta d\theta$. Finally, we can make the substitution

$$\begin{aligned}\int x^3\sqrt{9-x^2}\,dx &= \int (3\cos\theta)^3(3\sin\theta)(-3\sin\theta d\theta)\\ &= -3^5\int \cos^3\theta\sin^2\theta\,d\theta.\end{aligned}$$

Now we have a trigonometric integral that can be handled with the substitution $u=\sin\theta$ and $du=\cos\theta d\theta$.

$$\begin{aligned}\int x^3\sqrt{9-x^2}\,dx &= -3^5\int \cos^3\theta\sin^2\theta\,d\theta\\ &= -3^5\int \cos^2\theta\cdot u^2\,du\\ &= -3^5\int (1-\sin^2\theta)u^2\,du\\ &= -3^5\int (1-u^2)u^2\,du\\ &= -3^5\int (u^2-u^4)\,du\\ &= -3^5\left(\frac{1}{3}u^3-\frac{1}{5}u^5\right)+C\\ &= -3^5\left(\frac{1}{3}\sin^3\theta-\frac{1}{5}\sin^5\theta\right)+C\\ &= -3^5\left(\frac{1}{3}\left(\frac{1}{3}\sqrt{9-x^2}\right)^3-\frac{1}{5}\left(\frac{1}{3}\sqrt{9-x^2}\right)^5\right)+C\\ &= -3(9-x^2)^{\frac{3}{2}}+\frac{1}{5}(9-x^2)^{\frac{5}{2}}+C.\end{aligned}$$

3. The following questions ask you to discuss the use of the Midpoint Rule to approximate the value of $\int_0^6 \sin(2x)\,dx$.

(a) Estimate the value of the integral using the Midpoint Rule on three subintervals.

Solution: If we break the interval $[0,6]$ into three equal segments, each rectangle will have width 2. The midpoints of these intervals have x-coordinate 1, 3, and 5, respectively. Our approximation of the area becomes

$$\begin{aligned}\int_0^6 \sin x\,dx &\approx 2\sin(2)+2\sin(6)+2\sin(10)\\ &\approx 0.17172.\end{aligned}$$

(b) Give a good upper bound on $|E_M|$ when three subintervals are used. (Three subintervals is not very many, so your error will be quite large.)

Solution: Let $f(x) = \sin(2x)$. To determine the value of K, we first need to compute $f''(x)$.

$$f'(x) = 2\cos(2x)$$
$$f''(x) = -4\sin(2x)$$

Now we need to find a reasonable upper bound on the interval $[0, 6]$.

$$\begin{aligned} |f''(x)| &= |-4\sin(2x)| \\ &= |-4||\sin(2x)| \\ &= 4|\sin(2x)| \\ &\leq 4. \end{aligned}$$

We can therefore choose $K = 4$. (In fact, this is the smallest possible K since $\left|f''\left(\frac{\pi}{4}\right)\right| = 4$.) Finally, we have

$$\begin{aligned} |E_M| &\leq \frac{K(b-a)^3}{24n^2} \\ &= \frac{4(6-0)^3}{24 \cdot 3^2} \\ &= 4. \end{aligned}$$

(c) How many subintervals would be necessary to ensure $|E_M| \leq 0.0001$?

Solution: We must solve for n in

$$\begin{aligned} \frac{4(6-0)^3}{24 \cdot n^2} &\leq 0.0001 \\ \frac{4(6-0)^3}{24 \cdot 0.0001} &\leq n^2 \\ 360000 &\leq n^2 \\ 600 &\leq n. \end{aligned}$$

We must therefore use at least 600 subintervals.

4. For parts (a) and (b), evaluate the integral or demonstrate that it diverges. For part (c), use the comparison theorem to demonstrate convergence or divergence.

(a) $\displaystyle\int_0^\infty e^x\,dx$

Solution:

$$\begin{aligned}\int_0^\infty e^x\,dx &= \lim_{t\to\infty}\int_0^t e^x\,dx\\ &= \lim_{t\to\infty}\left(e^t-1\right)\\ &= \infty.\end{aligned}$$

The integral diverges.

(b) $\displaystyle\int_0^1 \frac{1}{x^3}\,dx$

Solution:

$$\begin{aligned}\int_0^1 \frac{1}{x^3}\,dx &= \lim_{t\to 0}\int_t^1 \frac{1}{x^3}\,dx\\ &= \lim_{t\to 0}\left(-\frac{1}{2}+\frac{1}{2t^2}\right)\\ &= \infty\end{aligned}$$

The integral diverges.

(c) $\displaystyle\int_1^\infty \frac{1-e^{-2x}}{x^2}\,dx$

Solution: None of the standard integration techniques seems to help here, so we will compare the integral to a simpler one. Since e^{-2x} is positive for all x, it follows that $\frac{1-e^{-2x}}{x^2}\le\frac{1}{x^2}$ for all x. Now,

$$\begin{aligned}\int_1^\infty \frac{1-e^{-2x}}{x^2}\,dx &\le \int_1^\infty \frac{1}{x^2}\,dx\\ &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx\\ &= \lim_{t\to\infty}\left(-\frac{1}{t}+1\right)\\ &= 1.\end{aligned}$$

By the Comparison Theorem, we conclude that $\int_1^\infty \frac{1-e^{-2x}}{x^2}\,dx$ is convergent.