Name: Austin Mohr
Please write *only* your name on the test sheet.
Place all work and answers on the blank sheets provided.
All items are equally weight unless otherwise noted.

1. The population of a small town grows according to the function

$$P(t) = 2000 \cdot 1.02^t,$$

where t is measured in years since 2000.

(a) What are the growth factor, growth rate, and instantaneous growth rate of this population?

(b) What is the doubling time of the population?

Solution: The function is presented in Ca^t form, where C is the initial value and a is the growth factor. The growth factor is therefore 1.02, which corresponds to a growth rate of 0.02 (or 2%). To determine the instantaneous growth rate, we need to convert to Ce^{kt} form.

$$\begin{aligned} 1.02 &= e^k \\ \ln(1.02) &= \ln(e^k) \\ 0.0198 &\approx k. \end{aligned}$$

The instantaneous growth rate is roughly 0.0198 (or 1.98%).

The doubling time is the amount of time required for the population to double. Since the initial value is 2,000, we can find the length of time required to reach 4,000.

$$\begin{aligned} 4000 &= 2000 \cdot 1.02^t \\ 2 &= 1.02^t \\ \ln(2) &= \ln(1.02^t) \\ \ln(2) &= t\ln(1.02) \\ \frac{\ln(2)}{\ln(1.02)} &= t \\ 35 &\approx t \end{aligned}$$

The population doubles roughly every 35 years.

2. Recall that the decibel rating of a sound is given by

$$B(I) = 10 \log \frac{I}{I_0},$$

where I is the intensity of the measured sound in $\mathrm{W/m^2}$ and $I_0 = 10^{-12}\ \mathrm{W/m^2}$.

(a) What is the decibel rating of a sound with intensity $50,000\ \mathrm{W/m^2}$?

(b) What is the intensity of a sound with decibel rating 90?

Solution: In part (a), we can evaluate the function at the known intensity.

$$B(50000) = 10\log\frac{50000}{10^{-12}} = 10\log\left(5\cdot 10^{16}\right) \approx 167$$

The rating of the sound is roughly 167 decibels.

In part (b), we know the decibel rating and must isolate the intensity I.

$$\begin{aligned} 90 &= 10\log\frac{I}{10^{-12}} \\ 9 &= \log\frac{I}{10^{-12}} \\ 10^9 &= 10^{\log\frac{I}{10^{-12}}} \\ 10^9 &= \frac{I}{10^{-12}} \\ 10^9\cdot 10^{-12} &= I \\ 0.001 &= I \end{aligned}$$

The intensity of the sound is $0.001\ \mathrm{W/m^2}$.

3. The population growth of a certain species of bird is limited by its nesting requirements and is modeled by the function

$$P(t) = \frac{11200}{1+27\cdot 1.85^{-t}},$$

where t is measured in years since 2000.

(a) What is the carrying capacity of the population (i.e., the theoretical maximum population)?

(b) What will the population be in 2015?

Solution: The carrying capacity of a logistic function is the numerator, which is 11,200 in this case. (A more sophisticated way is to consider what happens when t becomes very large. Since 1.85^{-t} models exponential decay, we know it tends toward 0 over the long term. This means that the entire denominator tends toward 1, so the fraction as a whole tends toward 11,200.)

The year 2015 corresponds to the value $t = 15$, so we can evaluate to determine the population.

$$P(15) = \frac{11200}{1 + 27 \cdot 1.85^{-15}} \approx 11170$$

The population in 2015 is roughly 11,170 birds.

4. A patient is administered 200 mg of a therapeutic drug. Once all 200 mg have been absorbed into the bloodstream, 25% of the drug is expelled from the body every 90 minutes.
   (a) Why is an exponential model a good choice for this scenario?
   (b) Construct a model of the remaining mass using the form $m(t) = Ca^t$, where t is measured in hours after complete absorption.

Solution: An exponential model is a good choice since expulsion of 25% per 90-minute interval can be modeled by repeated multiplication of 0.75 for each 90-minute interval.

To construct the model, begin with $m(t) = Ca^t$. Since the time $t = 0$ corresponds to the moment of complete absorption, the initial value C is 200. We may therefore update our function to be $m(t) = 200a^t$. To determine the growth factor, we need another point of data (other than the initial value). A convenient choice is to notice that there will be 150 mg present in the body after 90 minutes since $200 \cdot 0.75 = 150$. Since t is measured in hours rather than minutes, the data point is $(1.5, 150)$ rather than $(90, 150)$. We can use the data to solve for the growth factor.

$$\begin{aligned} 150 &= 200a^{1.5} \\ 0.75 &= a^{1.5} \\ 0.75^{\frac{1}{1.5}} &= \left(a^{1.5}\right)^{\frac{1}{1.5}} \\ 0.825 &\approx a \end{aligned}$$

The model is therefore $m(t) = 200 \cdot 0.825^t$.

5. A certain cell phone plan costs \$20 per month plus 5 cents for each minute used. The function

$$C(x) = 20 + 0.05x$$

therefore gives the monthly cost of talking for x minutes.

For every \$10 spent, the company awards the customer a point that can be used to redeem certain rewards. The function

$$P(x) = \frac{x}{10}$$

therefore gives the number of points earned for spending x dollars on the plan.

(a) Construct a function that gives the number of minutes used in a month as a function of the cost.

(b) Construct a function that gives the number of points earned as a function of the minutes used.

Solution: Before beginning, let's take a look at the domains and ranges of the functions we have. The function C takes minutes as input and gives cost as output. The function P takes cost as input and gives points as output.

For part (a), we need to make a function that takes cost as input and gives minutes as ouput. This is the reverse of what the function C does, so we want to determine the inverse of C. We begin by switching the role of x and y (that is, switching the roles of input and output) in the function and proceed to isolate y.

$$\begin{aligned} x &= 20 + 0.05y \\ x - 20 &= 0.05y \\ \frac{x-20}{0.05} &= y \end{aligned}$$

The desired function is therefore $C^{-1}(x) = \frac{x-20}{0.05}$. (Since $0.05 = \frac{1}{20}$, we could alternatively write $C^{-1}(x) = 20x - 400$.)

For part (b), we need a function that takes minutes as input and gives points as ouput. Since minutes are the input to the function C and points are the output of the function P, we hope that the composition $P(C(x))$ makes sense. It is indeed a valid composition, since the output of the function C is the same as the input of the function P (the cost of the plan). We can summarize all that information about the various domains and ranges with a diagram.

$$\text{minutes} \xrightarrow{C} \text{cost} \xrightarrow{P} \text{points}$$

Having established that the composition $P(C(x))$ is the right thing to consider, we can construct it.

$$P(C(x)) = P(20 + 0.05x) = \frac{20 + 0.05x}{10} = 2 + 0.005x.$$