Generating Functions

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1 The Generating Function

A generating function is an extremely powerful tool in combinatorics that compresses the information of an infinite sequence into a single equation.

1.1 Defining a Generating Function

Before we can begin to properly utilize a generating function, we must first properly define it. First and foremost, a generating function is a formal power series. A formal power series is a series of the form

$$\sum_{n\geq 0} b_n x^n$$

with $b_n \in \mathbb{R}$. While identical in form to a regular power series, functionally, a formal power series is quite different. When working with a formal power series, we give no thought to matters of convergence or to x as a number, or even a variable. In this way, the name "generating function" is indeed a misnomer, as we will never be using one as an actual function to map one set to another. Instead, as a formal power series, we are only interested in the ordered list of coefficients b_n. All of the relevant information contained in a generating function is stored in the coefficients, with the powers of x serving as indeterminates that work as placeholders for these coefficients.

1.2 Types of Generating Functions

The different forms of generating functions are as varied as their uses. This paper will focus on two types, the ordinary generating function and the exponential generating function. The type of generating function best suited for an individual problem is dependent on the structure of the problem; for example, outside of the baseline ordinary generating function, strictly combinatorial counting problems or problems involving ordering sets are best tackled with the exponential generating function. The reasoning for this usage will become clear when we examine the structure of the exponential generating function.

2 Ordinary Generating Functions

The most basic type of generating function is the ordinary generating function. The ordinary generating function of the sequence a_n takes the form

$$A(x) = \sum_{n \geq 0} a_n x^n.$$

The idea is to take our infinite sequence, a_n, multiply it by x^n, and sum from zero to infinity. This essentially strings our infinite sequence into the single formal power series that we're looking for, changing the sequence into a more workable form.

2.1 Forming an Ordinary Generating Function

To form our first ordinary generating function, we will examine the following problem: In how many ways can you split n dollars into ones, fives, and tens? If we let our sequence a_n be the the number of ways to split n dollars into ones, fives, and tens, then how can we construct our generating function $A(x) = \sum_{n \geq 0} a_n x^n$? We begin by first examining the question of how many ways you can split n dollars into ones. We'll let f_n be the number of ways to split n dollars into ones and construct the generating function $F(x)$. Obviously, for any n dollar amount, there is only one way to split it into ones, so the coefficient for each power of n in the formal power series $F(x)$ would be 1.

$$F(x) = \sum_{n \geq 0} f_n x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Now, the formula for the sum S of a geometric series states that

$$S = \sum_{n \geq 0} ar^n = a \left(\frac{1}{1-r} \right)$$

so we can simplify our generating function as follows:

$$F(x) = \sum_{n\geq 0} f_n x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$
$$F(x) = \frac{1}{1-x}.$$

Note that this summation formula normally carries the restriction that the absolute value of r must be less than 1; however, as we are working with a formal power series, we are unconcerned with convergence. As x is an indeterminate and not a variable, it is meaningless to think of it as a numerical value. Instead, we use this summation formula simply as a way to condense our infinite series into a much more compact form that contains all of the same information, namely the ordered set of coefficients $< 1, 1, 1, 1, 1, \cdots >$. This summation formula is used commonly when working with generating functions to change from the full formal power series form with coefficients to the more compact form and vice versa.
For the second part of the problem, we will let the series g_n be the number of ways to split n dollars into fives. For this problem, we see that there is one way to split n dollars into fives if n is divisible by 5, and zero ways if n is not divisible by 5. Thus, our formal power series $G(x)$ will have a coefficient of 1 for all terms where n is a multiple of 5, and a coefficient of 0 otherwise. So,

$$G(x) = \sum_{n\geq 0} g_n x^n = 1 + x^5 + x^{10} + x^{15} + x^{20} + \cdots$$
$$G(x) = \frac{1}{1-x^5}$$

Finally, we will let the series h_n be the number of ways you can split n dollars into tens. Mirroring our logic with g_n, we see that the generating function for h_n will have a coefficient of 1 for terms with values of n that are multiples of 10 and a coefficient of 0 otherwise. Thus we have

$$H(x) = \sum_{n\geq 0} h_n x^n = 1 + x^{10} + x^{20} + x^{30} + x^{40} + \cdots$$
$$H(x) = \frac{1}{1-x^{10}}$$

Now, we have three separate generating functions that still don't answer our original question. Observe, however, the product of the three generating functions that we created for our partial problems:

$$\begin{aligned}A(x) &= F(x)G(x)H(x)\\ &= (1 + x + x^2 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^{10} + x^{20} + \cdots)\end{aligned}$$

If we were interested in the coefficient of a certain term in this product of series (and indeed, as a formal power series, the coefficient is what interests us), we would have to choose a term from each of the three series that multiplied together to make the term we're interested in. For example, for the term with x^{33}, we could select x^3 from the first series $F(x)$, x^{10} from the second series $G(x)$, and x^{20} from the third series $H(x)$. Note that this is equivalent to splitting 3 dollars into ones, 10 dollars into fives, and 20 dollars into tens, essentially providing one method of splitting 33 dollars into ones, fives, and tens. There are other ways to choose terms that multiply together to make x^{33}; for example, we could choose x^8 from the first series, x^{15} from the second, and x^{10} from the third. Note that this is equivalent to splitting 33 dollars into 8 ones, 3 fives, and 1 ten. So we see that there exists a clear bijection between ways to choose terms from our three series that multiply to make a given x^n and ways to split n dollars into ones, fives, and tens. This bijection means that the coefficient of x^n in our product $A(x)$ is the number of ways to split n dollars into ones, fives, and tens, or exactly a_n. Thus, we see that

$$\begin{aligned}A(x) &= (1 + x + x^2 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^{10} + x^{20} + \cdots)\\ A(x) &= \sum_{n\geq 0} a_n x^n.\end{aligned}$$

Which shows that our generating function for a_n is $A(x)$.

$$A(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})}$$

2.2 Using a Generating Function

There are indeed a multitude of uses for generating functions. and one of the more interesting and useful applications is the ability to create an explicit function from a recursive formula. As an example of this, we will create an explicit formula for a very well known recursive function, the Fibonnacci sequence. Let

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2}.$$

If we take the ordinary generating function of the Fibonnacci sequence, we end up with

$$G(x) = \sum_{n \geq 0} F_n x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots$$

where the Fibonnacci sequence shows up as the coefficients for the terms in our series. Since these formal power series can be thought of as an ordered list of coefficients, the expression F_{n-1} can be thought of as this same ordered list, only shifted by one place, or one power of x. Similarly, F_{n-2} can be represented by the same ordered list of coefficients shifted by two places, or by x^2. Following this reasoning, we can make the following substitutions into our recursive formula for the Fibonnacci sequence:

$$\begin{aligned} F_{n-1} + F_{n-2} &= xG(x) + x^2 G(x) \\ &= x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots \end{aligned}$$

We see that this series is remarkably similar to our original formal power series expansion of $G(x)$, with a single x as the only term missing. Accounting for this, we have

$$\begin{aligned} G(x) &= xG(x) + x^2 G(x) + x \\ G(x) - xG(x) - x^2 G(x) &= x \\ G(x)(1 - x - x^2) &= x \\ G(x) &= \frac{x}{1 - x - x^2} \end{aligned}$$

and we have our generating function $G(x)$ for the Fibonnacci sequence. In order to extract an explicit formula, however, we have to put this generating function into a form with easily identifiable coefficients. To start this, we first factor out the denominator.

$$G(x) = \frac{x}{(1 - \varphi x)(1 + \frac{1}{\varphi} x)}.$$

Here, φ is $\frac{1+\sqrt{5}}{2}$, the golden ratio. At approximately 1.618, the golden ratio appears in many places in mathematics, architecture, and nature, and is intricately tied to the Fibonnacci sequence (as made clear by this example!).

Now, we apply partial fraction decomposition to obtain

$$G(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi x} - \frac{1}{1 + \frac{1}{\varphi} x} \right)$$

which contains two series as the terms inside the parentheses.

$$G(x) = \frac{1}{\sqrt{5}} \left(\sum_{n \geq 0} \varphi^n x^n + \sum_{n \geq 0} \frac{1}{\varphi^n} x^n \right)$$

In this form, it is quite simple to extract out the coefficient of x^n in the series. We see that the coefficient of x^n in $G(x)$ is

$$\frac{1}{\sqrt{5}} \left(\varphi^n + \frac{1}{\varphi^n} \right),$$

and since we know from our original definition of the ordinary generating function that

$$G(x) = \sum_{n \geq 0} F_n x^n,$$

we see that our coefficient of x^n is exactly the nth Fibonnacci number. Thus we have successfully created an explicit formula for the Fibonnacci sequence, namely

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n + \frac{1}{\varphi^n} \right).$$

3 Exponential Generating Functions

Exponential generating functions are similar in form and use to ordinary generating functions, with a slight difference in construction. An exponential generating function of a series a_n takes the form

$$E(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}.$$

This form, with the added division by $n!$, comes in handy when working with counting permutations, as the division by $n!$ essentially accounts for the number of ways to order n elements. The extra division by $n!$ is also useful when working with sequences that grow too fast for an ordinary generating function to have a closed form. We will show an example of both of these uses.

3.1 Counting with Exponential Generating Functions

A football coach has n players to work with. He splits them into two groups, has each member of the first group choose either a red, white, or black jersey, and then has each group line up. In how many ways can all of this be done?

In order to solve this problem, we will utilize a theorem we alluded to with our first example, the product rule for generating functions. The theorem can be stated as follows:

Theorem: Let a_n be the number of ways to build a certain structure on an n-element set and let b_n be the number of ways to build another structure on an n-element set. Let c_n be the number of ways to separate $[n]$ into disjoint subsets S and T such that $S \cup T = [n]$, then build the structure of a_n on S and the structure of b_n on T. Let $A(x), B(x)$, and $C(x)$ be the generating functions for the sequences a_n, b_n, and c_n respectively. Then $A(x)B(x) = C(x)$.

To solve this problem, assume the coach chooses k players to form the first group, and let a_k be the number of ways these players can select a red, white, or black jersey and line up. Since the number of ways k players can select a red, white, or black jersey is 3^k and the number of ways k players can line up is $k!$, we have $a_k = k!3^k$. Then our exponential generating function of a_k is

$$A(x) = \sum_{k \geq 0} k!3^k \frac{x^k}{k!} = \frac{1}{1-3x}.$$

Now, assume there are m people in the second group, and let b_m be the number of ways they can line up. Then $b_m = m!$ and our exponential generating function is

$$B(x) = \sum_{m \geq 0} m! \frac{x^m}{m!} = \frac{1}{1-x}.$$

If we let c_n be the number of ways the coach can split n players into two groups, have each member of the first group choose a red, white, or black jersey, and have each group line up, then the Product rule for generating functions implies that

$$\begin{aligned} C(x) = A(x)B(x) &= \frac{1}{1-3x}\frac{1}{1-x} \\ &= \sum_{n \geq 0} 3^n x^n \sum_{n \geq 0} x^n \end{aligned}$$

It follows that the coefficient of $\frac{x^n}{n!}$ is

$$c_n = n! \sum_{i=0}^{n} 3^i 1^{n-i} = n! \frac{3^{n+1} - 1}{2}.$$

3.2 Recursion with Exponential Generating Functions

Let $a_0 = 5$ and let $a_{n+1} = \frac{2(n+1)a_n + 5(n+1)!}{3}$. If we attempt to find an explicit formula with an ordinary generating function, the coefficients of our power series will become too large to give us a closed form. Instead, we will use exponential generating functions, with $G(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$. After some initial rearranging, we have

$$\frac{a_{n+1}}{n+1} = \frac{2}{3} a_n + \frac{5}{3} n!$$

and as per our exponential generating function, we will multiply by $\frac{x^n}{n!}$

$$a_{n+1} \frac{x^n}{(n+1)!} = \frac{2}{3} a_n \frac{x^n}{n!} + \frac{5}{3} x^n$$

and sum from 0 to infinity

$$\begin{aligned} \sum_{n \geq 0} a_{n+1} \frac{x^n}{(n+1)!} &= \frac{2}{3} \sum_{n \geq 0} a_n \frac{x^n}{n!} + \frac{5}{3} \sum_{n \geq 0} x^n \\ \frac{1}{x} \sum_{n \geq 1} a_n \frac{x^n}{n!} &= \frac{2}{3} \sum_{n \geq 0} a_n \frac{x^n}{n!} + \frac{5}{3} \sum_{n \geq 0} x^n. \end{aligned}$$

From our definition of $G(x)$, we can substitute the following:

$$\begin{aligned} \frac{1}{x}(G(x) - a_0 \frac{x^0}{0!}) &= \frac{2}{3} G(x) + \frac{5}{3} \frac{1}{1-x} \\ \frac{3}{x}(G(x) - 5) &= 2G(x) + \frac{5}{1-x} \end{aligned}$$

and solving for our generating function $G(x)$ yields

$$G(x) = \frac{5}{1-x} = \sum_{n \geq 0} 5n! \frac{x^n}{n!}$$

So our coefficient a_n of $\frac{x^n}{n!}$ implies that the explicit formula for our recursive function is

$$a_n = 5n!$$

4 Conclusion

With their ability to contain all of the infinite information about a sequence in a compact and manageable form, the generating function provides a unique and potent tool for dealing with otherwise intimidating problems. Ordinary and exponential generating functions are just two of many varieties, each with their own uses, but all utilizing the formal power series to create the proverbial and formidable numerical clothesline.