Ramsey Theory

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Abstract

The Ramsey Theorem guarantees that if the number of vertices of a complete graph is large enough, and we color the edges with two different colors, then we are guaranteed to find a monochromatic subgraph of a desired size. This argument can be extended if instead of two colors, we use an arbitrary number of colors. Other generalizations of the Ramsey Theorem will be discussed. With these facts taken together, we will observe that on a large enough set, complete disorder is impossible.

1 An Introduction

The bulk of information in this paper can be attributed to Miklós Bóna [1], unless otherwise noted. We shall motivate our discussion of Ramsey Theory with an introductory problem. Let there be six people in a room. We shall prove that either at least three people know each other, or at least three people do not know each other.

Denote the six people in the lobby by six vertices of a complete graph. Color the edge between two distinct vertices red if the the corresponding people know each other, and blue if the people do not know each other. Take any one vertex v. As v has 5 edges, which will be colored by two different colors, the Pigeon-Hole Principle implies that one of the colors red and blue will occur at least three times. Without loss of generality, assume that v has 3 red edges connected to it. Let x, y, and z be the other endpoints of these red edges. If each of x, y, and z are connected to each other by blue edges, then there exist three people that don't know each other, and we are done. Otherwise, at least one of the edges is red. Then the edge between those two points, and the two edges that connect back to v are red, and so a red triangle is formed, and then three people do know each other. Therefore, in any group of 6 people, either at least three people know each other, or three people do not know each other.

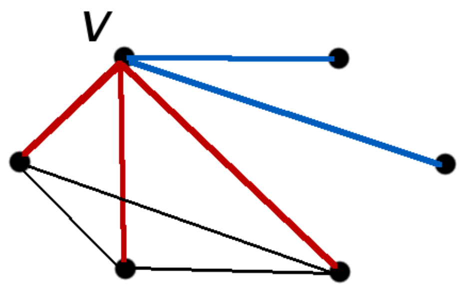


Figure 1: Pay particular attention to the black edges. They are a place-holder for the red and blue edges.

With this introductory problem out of the way, we now discuss the Ramsey Theorem.

2 The Ramsey Theorem

Theorem 1 (Ramsey Theorem)**.** *Let k and l be two positive integers, both of which is at least two. Then there exists a (minimal) positive integer R(k, l) so that if we color the edges of a complete graph with R(k, l) vertices red and blue, then this graph will either have a K_k subgraph with only red edges, or a K_l subgraph with only blue edges.*

Proof. We prove the Ramsey Theorem by induction. In particular, we shall prove the two following parts:

- Base Step: Show that $R(k, 2)$ and $R(2, l)$ exist for all k and l.
- The Inductive Step: Show that if $R(k, l$ - 1) exists, and also $R(k$ - 1, $l)$ exists, then $R(k, l)$ exists.

First we prove the base step by showing that $R(k, 2)$ and $R(2, l)$ exist for all k and l. Consider the number $R(k, 2)$. This is the minimal number of vertices on a graph required to guarantee either a K_k subgraph with all red edges, or a K_2 subgraph with all blue edges. Take a complete graph on k vertices. If all of the edges are red, then the whole graph is a monochromatic red K_k, and we are done. Otherwise, at least one of the edges is blue. Then the two vertices connected by a blue edge form a monochromatic blue K_2, and we are done again. Thus $R(k, 2) = k$. Similarly, $R(2, l) = l$. Thus the base step has been proven.

Now we verify the inductive step by showing:

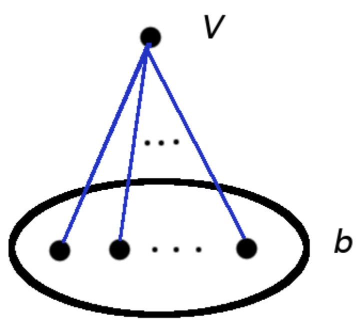
$$R(k,l) \leq R(k,l-1) + R(k-1,l).$$

In other words, we shall show that if there are $R(k,\ l$ - 1) + $R(k$ - 1, $l)$ vertices on a graph, then this is enough to guarantee a monochromatic red K_k, or a monochromatic blue K_l. Take a complete graph on $R(k,\ l$ - 1) + $R(k$ - 1, $l)$ vertices. Take any one vertex v, and note that v has degree $R(k,\ l$ - 1) + $R(k$ - 1, $l)$ - 1. Then either v has at least $R(k,\ l$ - 1) blue edges, or $R(k$ - 1, $l)$ red edges. Let us assume that v has at least $R(k,\ l$ - 1) blue edges adjacent to it. Let b denote the $R(k,\ l$ - 1)-element set of these other endpoints. By definition of $R(k,\ l$ - 1), the set b contains either a monochromatic red K_k, or a monochromatic blue $K_l - 1$, which can be completed to a K_l by adding the vertex v. Thus if v has at least $R(k,\ l$ - 1) blue edges adjacent to it, then there are at least $R(k,\ l)$ vertices on our graph. The same conclusion is reached if we assume v has $R(k$ - 1, $l)$ red vertices. Thus it holds that

$$R(k,l) \leq R(k,l-1) + R(k-1,l), \tag{1}$$

so the inductive step has been verified, and the Ramsey Theorem has been proven.

□



The Ramsey theorem guarantees us that some minimal positive integer $R(k,\ l)$ exists, but it doesn't tell us what the number is or how we might find it. We can however determine an upper bound on the $R(k,\ l)$.

Theorem 2. *Let k and l be positive integers larger than one. Then,*

$$R(k,l) \leq \binom{k+l-2}{k-1}.$$

Proof. First we show that this inequality holds for $R(2,l)$ and $R(k,2)$. Consider $R(2,l)$, and observe that

$$\binom{k+l-2}{k-1} = \binom{l}{1} = l,$$

and so it holds that $R(2,l) \leq \binom{k+l-2}{k-1}$. The same is true for $R(k,2)$.

Now we prove the inductive step by assuming this inequality holds for $R(k,l-1)$ and $R(k-1,l)$, and showing that it holds for $R(k,l)$. By equation (1) and the induction hypothesis, it holds that

$$R(k,l) \leq R(k,l-1) + R(k-1,l) \leq \binom{k+l-3}{k-1} + \binom{k+l-2}{k-2} = \binom{k+l-2}{k-1}$$

□

The following table [?] holds values of $R(r,s)$ for values of r and s ranging from 2 to 10. Where exact values are not known, the best known approximation is given.

r,s	1	2	3	4	5	6	7	8	9	10
1	**1**									
2	1	**2**								
3	1	3	**6**							
4	1	4	9	**18**						
5	1	5	14	25	**43–49**					
6	1	6	18	36–41	58–87	**102–165**				
7	1	7	23	49–61	80–143	113–298	**205–540**			
8	1	8	28	58–84	101–216	132–495	217–1031	**282–1870**		
9	1	9	36	73–115	126–316	169–780	241–1713	317–3583	**565–6588**	
10	1	10	40–42	92–149	144–442	179–1171	289–2826	331–6090	581–12677	**798–23556**

As our parameters k and l increase slightly, it becomes impossible to create all 2-colorings of a complete graph on $R(k,l)$ vertices. Consider a complete graph on 40 vertices. The number of edges is equal to the number of ways to choose two distinct vertices out of a set of 40. This is $\binom{40}{2}$ ways. Since each graph is a 2-coloring, there are $2^{\binom{40}{2}}$ 2-colorings of a complete graph on 40 vertices. This value is approximately $6.4 * 10^{234}$. Now if we were able to create one-billion different graphs per second, then the time required to create every possible 2-coloring would be about $1.5 * 10^{208}$ times the age of the universe.

3 Generalizations of the Ramsey Theorem

The Ramsey Theorem guarantees that order will be found on 2-colorings of a complete graph. But what if we want to color the edges of our graph with an arbitrary number of colors? The Ramsey Theorem can be generalized as follows.

Theorem 3. *Let $n_1, n_2, ..., n_k$ be positive integers, with k fixed. Then there exists a minimal positive integer $N = R(n_1, n_2, ..., n_k)$ so that if $n > N$, and we color all edges of $G = K_n$ with colors 1, 2, ..., k, then there will always be at least one index $i \in [k]$ so that G has a K_{n_i} subgraph whose edges are all of color i.*

Proof. We show that $R(n_1, ..., n_c) \leq R(n1, ..., n_{c-2}, R(n_{c-1}, n_c))$. Notice that the right-hand side of this equation only involves Ramsey numbers on $c-1$ colors, or on two colors. As we showed earlier in the proof of the Ramsey Theorem, the Ramsey number on two colors exists, so the number on the right-hand side of this equation must exist.

Now we prove the inductive step. Let there be $R(n1, ..., n_{c-2}, R(n_{c-1}, n_c))$ vertices on a graph. Imagine that the colors $c-1$ and c are the same color. So now our graph is $(c-1)$-colored. By definition of $R(n1, ..., n_{c-2}, R(n_{c-1}, n_c))$, the graph either has a monochromatic K_{c_i} for $i \in [c-2]$, or it contains a "monochromatic" $K_{R(n_{c-1}, n_c)}$. If the first case is true, then we are done. Otherwise, imagine now that $c-1$ and c are different colors. Then by definition of $R(n_{c-1}, n_c)$, this graph contains either a monochromatic $K_{n_{c-1}}$ of color $c-1$, or a monochromatic K_{n_c} of color c, and we are done again. □

Theorem 4 (Schur's Theorem)**.** *For every positive integer r, there exists a positive integer S, such that for every partition of the integers $\{1, ..., S\}$ into r parts, one of the parts contains integers x, y, and z such that $x + y = z$.*

Proof. Let $n = R(3, ..., 3)$, where $R(3, ..., 3)$ is the Ramsey number on r colors. Let $S = n$ and partition $\{1, ..., n\}$ into r parts. Let C denote the coloring of the integers in $[r]$.

Consider the complete graph K_n. Let x and y be two vertices and color their edge by the color $|x - y|$ in C. By definition of $R(3, ..., 3)$, K_n contains a monochromatic triangle with vertices $i > j > k$. Then $i - j, i - k$, and $j - k$ are of the same color c, and they belong to the same partition of $\{1, ..., n\}$. Let $x = i - j, y = j - k, z = i - k$. Then $x + y = (i - j) + (j - k) = i - k = z$.

□

The Ramsey Theorem can also be applied to geometry.

Theorem 5 (Erdös-Szekeres theorem)**.** *Let n be a positive integer. Then there exists a (minimal) positive integer $ES(ri)$ so that if there are $N \geq ES(n)$ points given in the plane, no three of which are collinear, then we can choose n points from them that form a convex n-gon.*

References

[1] Miklos Bona, *A Walk Through Combinatorics*, second ed., World Scientific Publishing Co., 2006.