Posets and the Möbius Function

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Abstract

The Möbius function and Möbius Inversion Formula can be used to define many properties of a poset. We will look at two applications of the Möbius function and Möbius Inversion Formula on a general poset.

Posets

Consider the situation of looking up flights for an upcoming trip. Imagine that there are 5 airlines offering a flight to the destination and each airline has a different price and flight time. Assume the following is each airline's offer:

A: 600 dollars; 9 hours 20 minutes
B: 650 dollars; 8 hours 40 minutes
C: 550 dollars; 9 hours 10 minutes
D: 575 dollars; 8 hours 20 minutes
E: 660 dollars; 9 hours 5 minutes.

It is clear to see that airline C has a better offer than airline A, but it is not clear whether airline A or airline D has the better offer. This is an example of a partially ordered set, or **poset**. In a poset, some, but not necessarily all pairs of elements are comparable. A set $P_{\leq}$ is a poset if $\leq$ is a relation on P such that the following hold:

1. $\leq$ is reflexive, that is, $x \leq x$ for all $x \in P$,
2. $\leq$ is transitive, that is, if $x \leq y$ and $y \leq z$, then $x \leq z$,
3. $\leq$ is antisymmetric, that is, if $x \leq y$ and $y \leq x$, then $x = y$.

Posets are represented using a **Hasse diagram**, which is a graph whose vertices are the elements of the set, if $x \leq y$ then x is above y on the graph, and there exists an edge between x and y if and only if there does not exist a z such that $x \leq z \leq y$. For the airline example, assume that $x \leq y$ if airline y has a shorter and cheaper flight than airline x, then the Hasse diagram is shown in Figure 1.

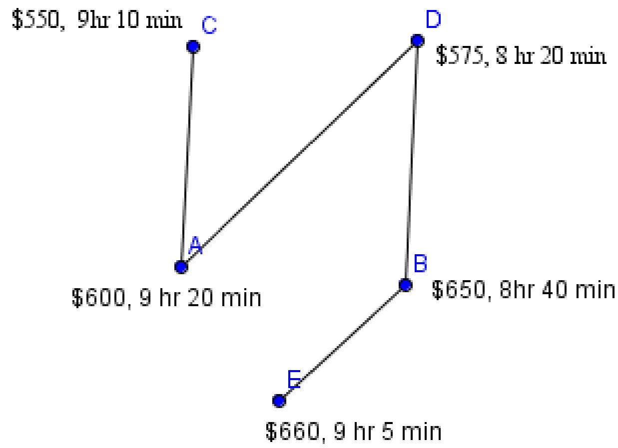


Figure 1: Hasse Diagram of Airline Offers

If P is a poset and there exists a subset $Q \subseteq P$ such that Q contains no incomparable elements, then Q is a **chain**. The length of a chain is the number elements minus one, or the number of edges in the Hasse diagram of the chain. For example, considering the 5 airline offers, airlines E, B, D form a chain of length 2 and airlines C, A form a chain of length 1.

If all chains of a poset P are finite, then P is a **locally finite** poset. Another way to define locally finite is in terms of the closed intervals of a poset P. Let $x, y \in P$ where P is a poset, and assume $x \leq y$. Then the set of all z such that $x \leq z \leq y$ is the **closed interval** between x and y and is denoted $[x, y]$. If all the closed intervals are finite, then P is locally finite.

Möbius Function

The **incidence algebra** $I(P)$ of a poset P is the set of all functions $f : Int(P) \to \mathbb{R}$, where $Int(P)$ is the set of all closed intervals of P. Multiplication in this algebra is the same as matrix multiplication. The **unit element**, u of $I(P)$ is defined to be

$$u = \delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

Note that u is equal to the identity matrix I if the (i, j) entry of a $n \times n$ matrix is equal to $\delta(x_i, x_j)$ for all x_i, x_j in a poset P. Therefore when matrix formalism is being used this element will be denoted using I.

Another useful function in $I(P)$ is the **zeta function**, which is defined as

$$\zeta(x,y) = \begin{cases} 1 & x \leq y \\ 0 & otherwise \end{cases}$$

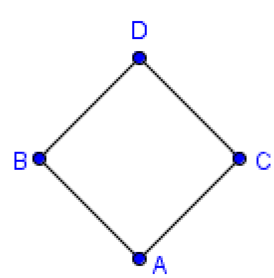
. When using matrix formalism the zeta function will be denoted using Z. Using these two functions, the number of chains of length k in a locally finite poset P can be calculated, as seen in the following lemma and proof.

Lemma: Let P be a locally finite poset. Let $[x, y] \in Int(P)$. Then the number of chains of length k that start at x and end in y is $(\zeta - \delta)^k(x, y)$.
Proof: (By Induction) *Base Case:* $k = 1$: Note that there is only one chain of length 1 possible between any x, y and it only exists if $x < y$. Then it must be shown that $(\delta - \zeta)(x, y) = 1$ if and only if $x < y$. Let $x, y \in P$. If and only if $x \leq y$, $\zeta(x, y) = 1$ and if and only if $x = y$, $\delta(x, y) = 1$. Thus $(\zeta - \delta)(x, y) = 1$ if and only if $x < y$ and $(\zeta - \delta)(x, y)$ is the number of chains of length 1 between x and y.
Inductive Step: Assume the number of chains between x and y of length $k - 1$ is given by $(\zeta - \delta)^{k-1}(x, y)$. Let $z \in [x, y]$. Then a chain of length k between x and y can be split into two chains, one of length $k - 1$ from z to y and one of length 1 from x to z. Summing over all $z \in [x, y]$ gives the total number of chains of length k from x to y,

$$\sum_{z\in[x,y]} (\zeta - \delta)(x, z)(\zeta - \delta)^{k-1}(z, y) = (\zeta - \delta)^k(x, y).$$

Thus the number of chains of length k from x to y is given by $(\zeta - \delta)^k(x, y)$. Q.E.D.

The **Möbius Function** $\mu(x, y)$ is defined as the inverse of the zeta function. Thus, $\mu(x, x) = 1$ by definition. If P is a locally finite poset and $[x, y] \in Int(P)$ then $\mu(x, y) = -\sum_{x\leq z\leq y} \mu(x, z)$. This formula is computed going from the bottom up when viewed as a Hasse diagram. An example of this calculation is shown below.
Example: Assume we have the Hasse Diagram shown below and calculate all values of the Möbius function.



$$\mu(A,A) = 1 \text{by definition.}$$

$$\mu(A,B) = -\sum_{A\le z\le B}\mu(A,z) = -\mu(A,A) = -1.$$

$$\text{Similarly,}\mu(A,C) = -1$$

$$\begin{aligned}\mu(A,D) &= -\sum_{A\le z\le D}\mu(A,z)\\ &= -(\mu(A,A)+\mu(A,B)+\mu(A,C))\\ &= -(1-1-1)\\ &= 1.\end{aligned}$$

The Möbius function of a locally finite poset P with $x_i, x_j \in P$ can also be calculated using the formula

$$\mu(x_i, x_j) = c_0 - c_1 + c_2 - c_3 + ...$$

where c_k is the number of chains of length k from x_i to x_j.
Proof: By the previously proved lemma, c_k is the (i,j) entry of the matrix $(Z-I)^k$. Thus $c_0 - c_1 + c_2 - c_3 + ...$ is the (i,j) entry of $\sum_k (-1)^k (Z-I)^k$.
$\sum_k (-1)^k (Z-I)^k = (I+Z-I)^{-1} = Z^{-1}$ by Taylor series expansion.
$\mu(x_i, x_j)$ is defined to be the (i,j) entry of Z^{-1}, thus

$$\mu(x_i, x_j) = c_0 - c_1 + c_2 - c_3 + ...$$

Möbius Inversion Theorem

A set $I \subseteq P$, where P is a poset, is an **ideal** if $x \in I$ and $x \le y$ implies that $y \in I$. If an ideal is generated by one element, that is $I = \{y | y \le x\}$, then I is a **principal ideal**.

The major application of the Möbius function is the **Möbius Inversion Formula**. The Möbius Inversion Formula can be generalized for any locally finite poset P, as seen below.

Möbius Inversion Formula: Let P be a poset in which each principal ideal is finite and let $f : P \to \mathbb{R}$ be a function. Let the function $g : P \to \mathbb{R}$ be defined by

$$g(y) = \sum_{x\le y} f(x).$$

Then,

$$f(y) = \sum_{x \leq y} g(x)\mu(x,y).$$

The Möbius Inversion Formula can be used to give the number of **connected graph**, or graph where for any two elements x, y a path between the two can be found, of $[n]$. A **connected graph** is one in which for any two elements x, y a path between the two can be found. There are $2^{\binom{n}{2}}$ graphs on $[n]$ since each of the $\binom{n}{2}$ pairs of vertices can either be connected or not. However, not all of these graphs are connected. The connected components of a simple graph, or a graph with at most 1 edge connecting each pair of vertices, on $[n]$ partition n such that if vertices are in the same component,they are in the same block. Call this underlying partition V.
Let H be any partition of $[n]$ and say that the blocks of H have size $c_1, c_2, ..., c_h$. There is no way to directly tell how many blocks of H have the previously defined underlying partition V. However the number of graphs having some underlying partition D such that $D \leq_{\Pi_n} H$,in other words D is some refinement of H, is easier to count. Their are $2^{\sum_{i=1}^{h}\binom{c_i}{2}}$ possible graphs with underlying partition D.
Let $f(H)$ be the number of all graphs on $[n]$ with underlying partition H and $g(H)$ be the number of all graphs with underlying partition D. Then $g(H) = 2^{\sum_{i=1}^{h}\binom{c_i}{2}}$ as previously shown. However,

$$g(H) = \sum_{D \leq_{\Pi_n} H} f(D)$$

since that is how it was previously defined. Applying the Möbius Inversion Formula to this equation,

$$f(H) = \sum_{D \leq_{\Pi_n} H} g(D)\mu_{\Pi_n}(D,H).$$

Let N be the one block partition of $[n]$, which is the connected graph of $[n]$. Substituting N in for H yields

$$\begin{aligned} f(N) &= \sum_{D \leq_{\Pi_n} N} g(D)\mu_{\Pi_n}(D,N) \\ &= \sum_{D \in \Pi_n} 2^{\sum_{i=1}^{d}\binom{d_i}{2}}(-1)^{d-1}(d-1)! \end{aligned}$$

where d is the number of blocks of D and $d_{-1}, D-2, d_3, ...$ are the sizes of the blocks in D [1].

References

[1] Miklos. Bona, *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*, third ed., World Scientific Publishing Co. Pte. Ltd., Singapore, 2013, With a Forweword by Richard Stanley.