The Probabilistic Method

Victoria Rosenboom

Nebraska Wesleyan University
vrosenbo@nebrwesleyan.edu

May 14, 2014

Abstract

We look at two different theorems involving tournaments, one using non-constructive proof and the other using linearity of expectation. Additionally, we use linearity of expectation to look at a "real life" example and determine the expected value of the number of fixed points in a randomly selected n-permutation.

1 Preliminary Definitions

We will use the standard terminology found in [2].

Let Ω be a finite set of outcomes of some sequence of trials, so that all these outcomes are equally likely. Let $A \subseteq \Omega$. Then Ω is called a *sample space*, and A is called an *event*. The ratio

$$P(A) = \frac{|A|}{|\Omega|}$$

is called the probability of A.

Let $X : \Omega \to \mathbb{R}$ be a random variable so that the set $S = \{X(u) | u \in \Omega\}$ is finite. Thus X only takes a finite number of values. Then the number

$$E(X) = \sum_{i \in S} i \cdot P(X = i)$$

is called the *expected value* or average of X on Ω.

2 Non-Constructive Proofs

A tournament on a set V of n players is an orientation $T = (V, E)$ of the edges of the complete graph on the set of vertices V. So we know that for every two distinct elements $x, y \in V$ either (x, y) or (y, x) is in E but not both. When we say (x, y) is in the tournament, we mean that player x beat player y. We also say that T has the property S_k if for every set of k players there is one player that beats them all. For example, let's look at a directed triangle, tournament $T_3 = (V, E)$, where $V = 1, 2, 3$ and $E = (1, 2), (2, 3), (3, 1)$. This tournament has the property S_1.

Theorem 1. *If $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$ then there is a tournament on n vertices that has the property S_k.*

Proof. We will follow the proof found in [1]. Let there be a random tournament on the set $V = \{1, ..., n\}$. For every fixed subset K of size k of V, let A_k be the event that there is no player (vertex) that beats all the members of K. Since we know that, for each fixed vertex $v \in V - K$, the probability that v beats all the members of K is 2^{-k}. So $1 - 2^{-k}$ gives us the probability that v does not beat all the members of K. Also, $n - k$ events correspond to the various possible choices of v will be independent. Thus we know that $\Pr[A_K] = (1 - 2^{-k})^{n-k}$ and it follows that,

$$Pr\left[\bigvee_{\substack{K \subset V \\ |K|=k}} A_K\right] \leq \sum_{\substack{K \subset V \\ |K|=k}} Pr[A_K] = \binom{n}{k}(1 - 2^{-k})^{n-k} < 1.$$

Consequently, we have shown that there is a tournament on n vertices that has the property S_k. □

Consider how difficult this would have been if we had tried to construct the various possible tournaments on set V, depending on the size of n.

3 Linearity of Expectation

For linearity of expectation, we'll use a similar theorem to that found in [2].

Theorem 2. *(1) Let X and Y be two random variables defined over the same space Ω. Then*

$$E(X + Y) = E(X) + E(Y).$$

(2) Let X be a random variable, and let c be a real number. Then

$$E(cX) = cE(x).$$

Let's look at how linearity of expectation can be used to derive the following theorems.

Theorem 3. *The expected value of the number of fixed points in a randomly selected n-permutation is 1.*

For an example of this theorem, we shall look at an example that shall forevermore be known as "The Hilbert Fauxtel." The front desk assigned all n of the rooms, created a keycard for each room and placed them in their respective paper holders with the room number on each one. When the n guests arrived, the system was down with no hope of turning back on. So they randomly gave out the room numbers. After all the keycards were all gone, the system turned back on. What is the expected number of guests who received their assigned room number?

Proof. Let G be the number of guests that get their assigned room, we want to find the expectation of G. But all we know about G is that the probability that a guest is assigned the proper room is $1/n$. There are many different probability distributions of room permutations with this property, so we dont know enough about the distribution of G to calculate its expectation directly. However, we can use the linearity of expectation.

Let G_i be an *indicator variable* for the event that the ith guest gets his or her assigned room. That is, $G_i = 1$ if the guest gets his or her assigned room, and $G_i = 0$ otherwise. The number of guests that get their assigned room is the sum of these indicator variables:

$$G = G_1 + G_2 + \cdots + G_n. \tag{1}$$

We know that these indicator variables are not mutually independent because if $n - 1$ guests receive their assigned rooms, then the last guest will also receive his or her assigned room. However, since we are going to apply the linearity of expectation, this doesn't matter.

Next, we know $\frac{1}{n} = PrG_i = 1 = E[G_i]$. Now we can take the expected value of both sides of Equation (1) and apply linearity of expectation to the fixed points G_i:

$$\begin{aligned} E[G] &= E[G_1 + G_2 + \cdots + G_n] \\ &= E[G_1] + E[G_2] + \cdots + E[G_n] \\ &= \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \\ &= n\left(\frac{1}{n}\right) \\ &= 1. \end{aligned}$$

So even though we dont how the rooms were randomly assigned, we know that on average, only one guest will be given his or her assigned room. □

Before looking at another application of linearity of expectation, it is important to understand that a *Hamiltonian path* is a path that includes all the vertices of a graph.

Theorem 4 (Szele, 1943)**.** *There is a tournament T with n players and at least $n!2^{-(n-1)}$ Hamiltonian paths.*

Proof. We follow the proof presented in [1]. Let H be the number of Hamiltonian paths in some random tournament T with n players. We shall prove that there are at least $n!2^{-(n-1)}$ Hamiltonian paths in T.

For each permutation σ, we will let H_σ be the indicator variable for that permutation producing a Hamiltonian path. A Hamiltonian path occurs when $(\sigma(i), \sigma(i+1)) \in T$ for all i, that is $1 \leq i < n$. We know that,

$$H = \sum H_\sigma$$

We know that it is equally likely that $(\sigma(i), \sigma(i+1))$ or $(\sigma(i+1), \sigma(i))$ will appear in T. Thus applying the Linearity of Expectation to the equation above,

$$\begin{aligned} E[H] &= \sum E[H_\sigma] \\ &= n! \cdot (\frac{1}{2})^{n-1} \\ &= n!2^{-n-1}. \end{aligned}$$

Consequently, some tournament has at least $n!2^{-n-1}$ Hamiltonian paths. □

Linearity of expectation is a great theorem with a variety of applications because it requires little knowledge about whether the variables involved are independent.

References

[1] Noga Alon and Joel H. Spencer, *The Probabilistic Method*, third ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., Hoboken, NJ, 2008, With an appendix on the life and work of Paul Erdős. MR 2437651 (2009j:60004)

[2] M. Bóna, *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*, third ed., University of Florida, USA, 2011, With a Foreword by Richard Stanley.